Master’s Thesis

A Survey of Thrackles

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Abstract

A thrackle is a graph drawing in which every pair of edges meets exactly once. This thesis characterizes some known thrackles and presents various approaches to Conway’s Thrackle Conjecture, which states that the number of edges of a thrackle cannot exceed the number of its vertices. Toward the end of the paper, this thesis begins a classification of trees that can be drawn as thrackles using the arcs of great circles on spheres. The last chapter of the thesis includes several original results. In service of a thorough description of thrackles, the thesis gives known limits on the edges of thrackles, the relationship between thrackles and design theory, and discusses the embeddings of thrackles on two-dimensional manifolds. We present both a pseudo-taxonomy of thrackleable graphs and a partial inventory of connected thrackleable graphs.
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1 Introduction

John Conway introduced the word *thrackle* to the world as a Scottish fishing term. As a teenager on vacation in Scotland with his family, Conway encountered a fisherman holding a tangled fishing line. As Conway told it, the fisherman called his line "thrackled." The term stuck with him and later, as a young mathematician, Conway used the word "thrackle" to describe a type of graph drawing. Since Conway introduced the term to the mathematical community, no one has yet been able to confirm that "thrackle" is a regional term, let alone one having anything to do with fishing. Despite its apocryphal origins, the thrackle and Conway’s conjecture live on. In fact, there is still an unclaimed cash prize offered for a proof that Conway’s Thrackle Conjecture is true (Mitchell and O’Rourke, 2001).

Before presenting original results, we discuss what is known about thrackles, including basic facts and attempts to prove the Thrackle Conjecture. In this chapter, we present basic facts about graphs and thrackles. Chapter 2 contains early results from Woodall, whose seminal work on thrackles began at the conference where Conway posed the Thrackle Conjecture. In Chapter 3, we present a brief review of some combinatorial attempts to prove the Thrackle Conjecture. In Chapter 4, we discuss graph types in which the Thrackle Conjecture is true. Finally, in Chapter 5, we discuss thrackles on surfaces, in particular spheres. There, we expand on previous research to classify great circle thrackleable trees. In exploring the literature about thrackles, we reveal a pseudo-taxonomy of thrackleable graphs. In an appendix, we provide a partial list of graphs that are known to be thrackleable.
1.1 Properties of Graphs

Let us begin the discussion of thrackles with some basic definitions and known facts. For the remainder of the paper, let $G$ be a finite graph with vertex set $V(G)$, edge set $E(G)$, and region set $R(G)$. For consecutive edges $e_1, e_2, \ldots, e_k$, let $e_1 = \{v_0, v_1\}$, $e_2 = \{v_1, v_2\}$, \ldots, $e_k = \{v_{k-1}, v_k\}$.

**Definition 1.1.** The **degree** of a vertex in $v \in V(G)$, $\deg(v)$, is the number of edges incident to $v$. Similarly, the **degree** of region $r$, $\deg(r)$, is the number of edges in its boundary. If an edge appears twice in the boundary of $r$, it is counted twice in $\deg(r)$.

**Theorem 1.2.** Let $G$ be a graph containing $m$ edges. The sum of the degrees of the vertices of $G$ is $2m$.

**Proof.** Every edge in $G$ is incident to two vertices, so every edge is counted in the degree of two vertices. Therefore,

$$\sum_{v \in V(G)} \deg(v) = 2m.$$ 

The degrees of the vertices can be used to categorize or characterize graphs. Graphs can be grouped based on their **degree sequences**.

**Definition 1.3.** The **degree sequence** of a graph $G$ is an ordered sequence of integers, each representing the degree of a vertex in $G$.

For example, the degree sequence of a 3-cycle is $(2, 2, 2)$ and the degree sequence of a path of length three is $(1, 1, 2, 2)$. Note that a degree sequence may be non-increasing or non-decreasing.

1.2 The Thrackle Conjecture

Rather than consider graphs as only abstract sets, it is often useful to give them visual representation in two dimensions.
Definition 1.4. A drawing of $G$ is a representation of $G$ in the plane such that $V(G)$ corresponds to a set of points and $E(G)$ corresponds to a set of arcs between two points.

Definition 1.5. A thrackle is a drawing of a simple, undirected graph in which

1. no curve crosses itself and

2. every pair of curves intersects at exactly one point

If a graph can be drawn as a thrackle, it is called thrackleable.

For example, the 3-cycle and 3-path, in Figures 1 and 2 are both thrackleable.

![Figure 1: $C_3$](image1)

![Figure 2: Planar and Thrackle Drawings of 3-Path](image2)

Conway’s Thrackle Conjecture is an attempt to show exactly which graphs are thrackleable. Despite its simplicity, it remains an open question.

Conjecture 1.6 (Conway’s Thrackle Conjecture). Every thrackleable graph contains at least as many vertices as edges.

While we do not yet know which graphs are thrackleable, in general, we do have some guidance for constructing a thrackleable graph from an existing thrackleable graph. The following two widely known theorems are basic results about thrackles that are important to most other results about thrackles.
Theorem 1.7 (Woodall, 1971). Every subgraph of a thrackleable graph is thrackleable.

Proof. Toward a contradiction suppose that $G$ is a thrackleable graph with non-thrackleable subgraph $H$. Since $H$ is not thrackleable, there is at least one pair of edges in $H$ that cannot be drawn to meet exactly once at either a crossing or a vertex. Yet $H \subseteq G$, so this pair of edges forces $G$ to fail to be thrackleable. \qed

Lemma 1.8. If $G$ is a thrackleable graph, then $G$ with a leaf attached is thrackleable.

Proof. Let $G$ be a finite, thrackleable graph on $n$ vertices. Choose a vertex and label it $v$. Let us show that $G$ with a leaf attached at $v$ is thrackleable. Assign the label $v'$ to an arbitrary vertex adjacent to $v$. Consider a tubular neighborhood $U$ of the interior of \{ $v,v'$ \} which contains only \{ $v,v'$ \} and its proper crossings. Consider also a neighborhood $V$ of $v'$ that contains only $v'$ and edges incident to it.

Attach a leaf to $v$. Draw $\{ w,v \}$ so that in $U$ its interior never touches $\{ v,v' \}$, but crosses all the edges that cross $\{ v,v' \}$. Moreover, in $V$ draw $\{ w,v \}$ so that it crosses all edges incident to $v'$ except $\{ v,v' \}$. This drawing is a thrackle. \qed

Although (perhaps because) the Thrackle Conjecture has not yet been proved, it has spurred a substantial body of research. Here, we will examine some of the earliest research about thrackles from Douglas Woodall, one of Conway’s colleagues at Cambridge. We will also explore more recent approaches to the Thrackle Conjecture, including several computational proof attempts. We will finish by looking at indirect approaches to the Thrackle Conjecture, which pose other conjectures whose proofs would imply that Conway was correct. For example, if Li, Daniels, and Rybnikov’s (2007) 1-2-3 Conjecture is correct, then the Thrackle Conjecture is true. Moreover, if every thrackle can be drawn on the sphere using the arcs of great circles, then the Thrackle Conjecture is true.
2 Woodall’s Theorem

The first major characterizations of thrackles came from the 1969 conference where John Conway stated the Thrackle Conjecture. Douglas Woodall, one of Conway’s colleagues, published a number of simple results about thrackles, upon which all current and future research on the subject builds. Chief among them is the following theorem, which we prove in this section.

**Theorem 2.1** (Woodall, 1971). *If the Thrackle Conjecture is true, then a finite simple graph is thrackleable if and only if it contains at most one odd cycle and no 4-cycle, and each of its connected components either is a tree or contains exactly one n-cycle, for \( n \geq 3 \).*

2.1 N-Cycle Thrackles

To prove Theorem 2.1, we need several definitions and lemmas. First, we describe \( n \)-cycle thrackles.

**Theorem 2.2** (Woodall, 1971). *Every \( n \)-cycle is thrackleable for \( n \geq 3 \), \( n \neq 4 \).*

To prove Theorem 2.2, we start by demonstrating that the 4-cycle is not thrackleable. We cite the Jordan Curve Theorem.

**Theorem 2.3** (Jordan Curve Theorem). *If \( J \) is a simple closed curve, then \( \mathbb{R}^2 \setminus J \) has two components, one on the exterior of \( J \) and one on the interior of \( J \).*

**Lemma 2.4.** *The 4-cycle is not thrackleable.*

*Proof.* Let \( V(C_4) = \{v_0, v_1, v_2, v_3\} \), where \( v_i \) is adjacent to \( v_{i+1} \) for \( i \in \mathbb{Z}_4 \). Let \( e_i = \{v_i, v_{i+1}\} \). Cross \( e_0 \) and \( e_2 \) and call their crossing \( p \). Then the regions bounded by \( [v_0, p, v_3] \) and \( [v_1, p, v_2] \) are two triangles with shared point \( p \). To draw \( C_4 \) as a thrackle, it is necessary to cross \( e_1 \) and \( e_4 \), as in Figure 3. By the Jordan Curve Theorem, crossing \( e_1 \) and \( e_3 \) places either \( v_1 \) or \( v_2 \) in the interior of \( [v_0, p, v_3] \). Without loss of generality,
place \( v_1 \) in the interior of \([v_0, p, v_3]\). Then \( e_2 \) is adjacent to \( e_3 \) and crosses \( e_3 \). Thus \( C_4 \) is not thrackleable.

\[
\begin{array}{c}
\begin{array}{c}
v_2 \\
v_1 \\
v_0 \\
v_3
\end{array}
\end{array}
\quad \rightarrow 
\begin{array}{c}
\begin{array}{c}
v_3 \\
v_1 \\
v_0 \\
v_2
\end{array}
\end{array}
\]

Figure 3: 4-cycle

Now we turn to thrackleable \( n \)-cycles. First we define a type of thrackle called a musquash. Then we define the straight-line thrackle to show that all odd cycles are thrackleable.

**Definition 2.5.** A **musquash** is an \( n \)-cycle thrackle with circular symmetry determined by the order of its edges’ crossings. In musquash \( M \), let edge \( e_i \) cross edges with indices \( k_{i,1}, \ldots, k_{i,n-3} \) in that order, from \( v_{i-1} \) to \( v_i \). Then from \( v_{j-1} \) to \( v_j \), \( e_j \) crosses the edges with indices \( k_{j,1}, \ldots, k_{j,n-3} \) in that order, so that for a given \( l \), \( k_{j,l} - k_{i,l} = j - i \), modulo \( n \).

\[
\begin{array}{c}
\begin{array}{c}
v_4 \\
v_2 \\
v_0 \\
v_1 \\
v_3
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
v_5 \\
v_4 \\
v_3 \\
v_2 \\
v_1
\end{array}
\end{array}
\]

Figure 4: Musquashes of the 5-cycle (left) and 6-cycle (right)
Definition 2.6. A straight-line thrackle is a thrackle whose arcs are drawn as straight line segments.

Lemma 2.7. Every odd cycle can be drawn as a musquash that is a straight-line thrackle.

Proof. Let \( \{v_0, v_1, \ldots, v_{n-1}\} \) be the vertex set for \( C_n \), where \( v_i \) and \( v_{i+1} \) are adjacent for all \( i \in \mathbb{Z}_n \). Consider the unit circle. Fix some \( \theta \in [0, 2\pi) \). Without loss of generality, place \( v_0 \) at \((1, \theta)\), using polar coordinates. Then place every vertex \( v_i \) at \((1, \theta + \frac{i\pi}{n})\) if \( i \) is even or at \((1, \theta + \pi + \frac{i\pi}{n})\) if \( i \) is odd. Thus the counterclockwise ordering of the vertices is \( v_0, v_2, \ldots, v_{n-1}, v_1, v_3, \ldots, v_{n-2} \), as in Figure 5. For every \( i \in \mathbb{Z}_n \), represent \( e_i \) by a straight line segment. Then \( e_i \) and \( e_{i+1} \) are adjacent, but are separated by an angle of \( \frac{2\pi}{n} \), so they do not cross, although they share the vertex \( v_{i+1} \).

![Figure 5: Odd Cycle Musquash](image)

Now we show by induction that the ordering of the vertices forces non-adjacent edges to cross. Consider \( n = 3 \). Since all three edges of \( C_3 \) are adjacent, the claim is vacuously satisfied. Suppose that the claim holds for \( n \). Let us show that it holds for \( n + 2 \). Let \( \theta = \frac{3\pi}{2} \), as in Figure 5. Delete \( e_n \). Insert \( v_n \) between \( v_{n-2} \) and \( v_0 \) and insert \( v_{n+1} \) between \( v_{n-1} \) and \( v_1 \) and arrange the vertices on the circle so that the distance between vertices on the circle is at least \( \frac{2\pi}{n+2} \). Draw \( e_n \) as \( \{v_{n-1}, v_n\} \), \( e_{n+1} \) as \( \{v_n, v_{n+1}\} \), and \( e_{n+2} \) as \( \{v_0, v_{n+1}\} \). Then \( v_n \) is below odd \( v_i \) for \( i < n \) and to the left of even \( v_i \) for \( i \leq n + 1 \). On the other hand, \( v_{n+1} \) is to the right of odd \( v_i \) for \( i \leq n \) and above even \( v_i \) for \( i < n + 1 \). Thus, \( e_i \) crosses \( e_j \) for \( e_i \) non-adjacent to \( e_j \) and \( i \neq j \). \( \square \)
Note that for an odd $n$-cycle inscribed on a circle to be drawn as a thrackle, the distance between vertices need not be $\frac{2\pi}{n}$. We demonstrate this fact in Proposition 2.8.

**Proposition 2.8.** Let $C$ be an odd $n$-cycle with consecutive vertices labeled with consecutive indices in $\mathbb{Z}_n$. Fix an orientation on the circle. Place the even-indexed vertices in ascending order, starting from $v_0$, on the circle. Place the odd-indexed vertices in ascending order, starting from $v_1$, so that $v_1$ follows $v_{n-1}$ and $v_{n-2}$ precedes $v_0$. Draw the edges of $C$ as line segments between vertices with consecutive indices. This drawing is a straight-line thrackle.

**Proof.** Suppose that $n = 3$. Since all vertices in $C$ are adjacent, the claim holds.

Now suppose that the claim holds for $n = k - 2$. We will show that it holds for $k$. Fix a circle and draw a thrackle of a $(k - 2)$-cycle on it, with vertices labeled as in the claim. Insert $v_{k-2}$ between $v_{k-4}$ and $v_0$; then insert $v_{k-1}$ between $v_{k-3}$ and $v_1$. Note that $\{v_0, v_{k-3}\}$ crosses every other edge, except $\{v_0, v_1\}$ and $\{v_{k-4}, v_{k-3}\}$. Therefore, any edge drawn between the arc bounded by $v_{k-4}$ and $v_2$ and the arc bounded by $v_{k-3}$ and $v_1$ crosses all the edges crossed by $\{v_0, v_{k-3}\}$.

Delete the edge $\{v_0, v_{k-3}\}$ and draw $\{v_{k-3}, v_{k-2}\}$. Because $v_{k-2}$ is between $v_2$ and $v_{k-4}$, the new edge $\{v_{k-3}, v_{k-2}\}$ crosses every edge that was previously crossed by $\{v_0, v_{k-3}\}$. Moreover, since $v_{k-2}$ is between $v_{k-4}$ and $v_0$, drawing the edge $\{v_{k-3}, v_{k-2}\}$ forces it to cross $\{v_0, v_1\}$. Similarly, since $v_{k-1}$ is between $v_{k-3}$ and $v_1$, $\{v_{k-2}, v_{k-1}\}$ drawing it forces it to cross all previously drawn edges aside from $\{v_{k-3}, v_{k-2}\}$. Finally, draw $\{v_0, v_{k-3}\}$, which crosses every edge in $C$ besides $\{v_0, v_1\}$ and $\{v_{k-2}, v_{k-1}\}$. We demonstrate this process, in part, in Figure 6.
We turn now to even-cycle thrackles. Many even cycles are thrackleable through a Conway doubling. We define this process using the idea of a small neighborhood.

**Definition 2.9.** A small neighborhood of vertex $v$ is an open set containing $v$ and intersecting the interiors of only the edges incident to $v$. A small neighborhood of edge $e$ is a union of the neighborhoods of the interior points of $e$, which include interior points of only $e$ and the edges that cross $e$.

**Definition 2.10.** Let $G$ be an $n$-cycle, with consecutive vertices labeled with consecutive indices in $\mathbb{Z}_n$. Draw $G$ as a thrackle. Let $U_i$ be a small neighborhood of $v_i$. Delete all edges in $G$. For all $i \in \mathbb{Z}_n$, replace $v_i$ with a pair of disjoint vertices $v_i$ and $v_{i+n}$ in $U_i$. For consecutive indices $i$ and $i+1$, let $i$ be drawn inside(outside) the former boundary of $G$ and $i+1$ outside(inside) the former boundary of $G$. Draw edges between $v_i$ and $v_{i+1}$ for $i \in \mathbb{Z}_{2n}$ in loops as below. This drawing is a Conway doubling of $G$ (see Figure 7).
Theorem 2.11 (Woodall, 1971). Let $G$ be the graph resulting from a Conway doubling on $n$-cycle $C$. If $n$ is odd, then $G$ is a cycle of length $2n$. If $n$ is even, then $G$ is the vertex-disjoint union of two $n$-cycles.

Proof. Suppose that $n$ is odd. Fix $v_0$ with respect to the boundary of $C$, say inside. Then $v_{n-1}$ is also inside the boundary of $C$, so there is no edge $\{v_0, v_{n-1}\}$. Instead, there is a path $v_0v_1 \ldots v_{n-1}v_nv_{n+1} \ldots v_{2n-1}v_0$, which is a $2n$-cycle. Now suppose that $n$ is even. Again, fix $v_0$ inside the boundary of $C$. Then $v_{n-1}$ is outside the boundary of $C$, so $G$ contains the edge $\{v_0, v_{n-1}\}$. Thus, $G$ contains the closed path $v_0v_1 \ldots v_{n-1}v_0$. By hypothesis, $v_n$ is outside the boundary of $C$. Consequently, $v_{2n-1}$ is inside the boundary of $C$ and $G$ contains the edge $v_n, v_{2n-1}$ and the closed path $v_nv_{n+1} \ldots v_{2n-1}v_n$. Therefore, $G$ consists of two vertex-disjoint $n$-cycles.

The previous theorem demonstrates the thrackleability of even $n$-cycles where $n$ is the double of an odd integer. To show that all even cycles are thrackleable, we show that they are shackleable. In this way we can complete the proof of Theorem 2.2.

Definition 2.12. Let $G$ be a bipartite graph with independent sets of vertices $V_1(G)$ and $V_2(G)$. Let $U_1$ and $U_2$ be disjoint neighborhoods of $V_1(G)$ and $V_2(G)$, respectively. A shackle of $G$ is a thrackle of $G$ in which every edge is drawn as a straight-line segment between $U_1$ and $U_2$ (though not necessarily a straight-line segment within each neighborhood). A graph that can be drawn as a shackle is shackleable.
Lemma 2.13. Every even $n$-cycle is shackleable for $n \geq 6$.

Proof. Let $n = 6$. Then $C_n$ is shackleable, as shown in Figure 9. Now suppose that $C_n$ is shackleable for $n = k$. We will show that $C_n$ is shackleable for $n = k + 2$.

Begin with a shackle drawing of $C_k$. Label the two sets of independent vertices of $C_k$ $L$ and $R$ and let $W_L$ and $W_R$ be disjoint neighborhoods of $L$ and $R$, respectively. Suppose that the vertices with odd indices are in $L$ and those with even indices are in $R$. To demonstrate that $C_{k+2}$ is a shackle, we will replace one edge of $C_k$ with a 3-path, as in Figures 10 and 11. Without loss of generality, consider $e_{k-1}$. Let $V_{k-1}$ and $V_0$ be small neighborhoods of $v_{k-1}$ and $v_0$, respectively. Let $U$ be small neighborhood of $e_{k-1}$.

Delete $e_k$. Draw an arc incident to $v_{k-1}$ within $U \setminus (W_L \cup W_R)$ so that it crosses all edges except $e_1$ within $U$, then draw it crossing $e_1$ once outside $W_R$. Terminate this newly drawn edge with vertex $v_k$, between $v_0$ and $v_{k-2}$ and label it $e_{k-1}$. Now draw an arc incident to $v_k$ that crosses only $e_1$ in $V_1 \cap W_L$ and crosses every edge except $e_k$. 

Figure 8: Shackle Diagram

Figure 9: Shackle of $C_6$
Figure 10: Edge Replacement in an Even Cycle

Figure 11: Edge Replacement in a 6-cycle to Form an 8-cycle

and $e_{k-1}$ in $U$, in which it is a straight-line segment. Then draw it crossing $e_{k-1}$ in $V_{k-1} \cap W_R$ and terminate this edge at vertex $v_{k+1}$. Call this edge is $e_{k+1}$. Finally, draw an arc incident to $v_{k+1}$ so that it crosses $e_{k-1}$ in $V_{k-1} \cap W_R$. Then continue drawing this arc in $U$ so that it crosses every edge except $e_{k-1}$. Terminate this arc at $v_0$ and label it $e_{k+2}$. This graph is a $(k+2)$-cycle drawn as a shackle. This concludes the proof of Theorem 2.2.

Now we turn to the final part of Theorem 2.1 that deals with specifically $n$-cycles.

**Lemma 2.14.** The union of two disjoint odd cycles is not thrackleable.

**Proof.** Let $C_m$ and $C_n$ be two odd cycles. Let the vertices and edges of $C_n$ be labeled $v_i$ and $e_i$, for $i \in \mathbb{Z}_n$, where $e_i = \{v_i, v_{i+1}\}$. Without loss of generality, suppose that $v_0$ is on the exterior of $C_m$. Toward a contradiction, suppose that $C_m \cup C_n$ is thrackleable. Then in crossing all edges of $C_m$, $e_1$ terminates at $v_1$ on the interior of $C_m$. Because $n$ is odd, $e_{n-1}$ also terminates at $v_{n-1}$ on the interior of $C_m$. Yet $e_n$ terminates at $v_0$ on the interior of $C_m$ after crossing all edges of $C_m$. Clearly, this is not possible. Therefore, $C_m \cup C_n$ is not thrackleable. 

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2.2 Tree Thrackles

Now we turn to the thrackleability of trees. The notion of a shackle gives us a tool to show that trees are always thrackleable. This in turn provides part of the forward portion of Theorem 2.1.

**Lemma 2.15** (Woodall, 1971). *Every finite tree can be shackled.*

*Proof.* Let $T$ be a finite tree. Note that every tree can be constructed from a 1-path and the addition of successive leaves. Since $T$ is bipartite, it contains two independent sets of vertices, $L$ and $R$. Let $W_L$ and $W_R$ be neighborhoods of $L$ and $R$, respectively.

Let us show that $T$ can be shackled by induction on $|V(T)| = n$. Suppose that $n = 1$. Then $T$ is vacuously shackleable. Now suppose that $T$ is shackleable for $n = k$. Arrange the vertices of $T$ in a shackle drawing, such as the one in Figure 13. Consider $v \in T$. Without loss of generality, suppose that $v \in L$. Let $U$ be a small neighborhood of some edge incident to $v$, say $e$. Draw an arc from $v$ through $U$ to $R$, terminating at a leaf $w$. 

![Figure 12: Shackle Diagram](image)

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Figure 13: Shackle with Leaf Attached

Place \( v_k \) directly above the vertex \( v_j \) adjacent to \( v_i \) and below all others above \( v_j \) in \( R(T) \). If \( i = 0 \), then draw \( \{v_i, v_k\} \) as a straight-line segment. Otherwise, if \( i \neq 0 \), consider a small neighborhood \( U \) of \( \{v_i, v_j\} \) and a small neighborhood \( V \) of \( v_j \). Draw \( \{v_i, v_k\} \) in \( U \) so that it never crosses \( \{v_i, v_j\} \), but crosses every edges that \( \{v_i, v_j\} \) crosses. Then within \( V \), allow \( \{v_i, v_k\} \) to cross the edges incident to \( v_j \). This drawing is a shackle.

Let us conclude the discussion of shackles with two final lemmas.

**Lemma 2.16** (Woodall, 1971). *If \( G \) is a shackleable graph, then \( G \) with a leaf attached is shackleable.*

*Proof.* Let \( G \) be a finite, shackleable graph of order \( n \). Assign the labels \( L \) and \( R \) to the two independent sets of vertices of \( G \). Let \( W_L \) and \( W_R \) be disjoint neighborhoods of \( L \) and \( R \), respectively. Choose a vertex and label it \( v \). Without loss of generality, suppose \( v \in L \). Assign the label \( v' \) to an arbitrary vertex in \( R \) adjacent to \( v \). Let \( V \) and \( V' \) be small neighborhoods of \( v \) and \( v' \), respectively, and let \( U \) be a small neighborhood of \( \{v, v'\} \).

Draw an arc, that is disjoint from \( \{v, v'\} \), from \( v \) through \( U \) to a point in \( V' \setminus \{v'\} \). Label the endpoint of this arc \( w \). Ensure that \( \{v, w\} \) is a straight-line segment in \( U \setminus (W_L \cap W_R) \). In \( V' \), cross all edges incident to \( v' \) aside from \( \{v, v'\} \).
Lemma 2.17. The finite union of shackleable graphs is a shackleable graph.

Proof. Let $G_1, G_2, \ldots, G_k$, be shackleable graphs, with $L_i$ and $R_i$ as the pair of independent sets of vertices of $G_i$. Consider vertical lines $\gamma_L$ and $\gamma_R$ in $\mathbb{R}^2$ and $k$ disjoint open sets containing segments of each. Without loss of generality, label the bottom-most neighborhood of $\gamma_L U_1$. Label the remaining neighborhoods of segments of $\gamma_L$ with ascending indices from the bottom, $U_2, \ldots, U_k$. Label the top-most neighborhood of $\gamma_R V_1$ and assign the labels $V_2, \ldots, V_k$ in ascending order to open sets below $V_1$ on $\gamma_R$. Arrange the vertices of $L_i$ in $U_i$ and the vertices of $R_i$ in $V_i$ for $1 \leq i \leq k$.

Given this configuration of vertices, draw $G_1, \ldots, G_k$ independently as shackles so that their edges are straight-line segments in $\mathbb{R}^2 \setminus (\bigcup_{i=1}^k U_i \cup \bigcup_{i=1}^k V_i)$ and the edges of $G_i$ are otherwise restricted to $U_i$ and $V_i$. Since the edges of $G_i$ and $G_j$ cross in $\mathbb{R}^2 \setminus (\bigcup_{i=1}^k U_i \cup \bigcup_{i=1}^k V_i)$ for $i \neq j$, this drawing of $\bigcup_{i=1}^k G_i$ is a shackle.

Figure 14: Disjoint Union of Shackles
2.3 Proof of Woodall’s Theorem

The preceding results have given us the tools necessary to prove Woodall’s Theorem. We have already shown, assuming the Thrackle Conjecture is true, that the forward direction of the theorem is true by contrapositive. Now we show that the backward direction is true.

**Theorem** (Woodall, 1971). *If the Thrackle Conjecture is true, then a finite simple graph is thrackleable if and only if it contains at most one odd cycle and no 4-cycle, and each of its connected components either is a tree or contains exactly one n-cycle, for n ≥ 3.*

**Proof.** Suppose that the Thrackle Conjecture is true.

Let graph *G* be thrackleable. By Lemmas 2.14, 2.4, and 1.7, no subgraph of *G* is a 4-cycle and no more than one connected component of *G* contains an odd cycle. Since the Thrackle Conjecture holds, each connected component contains no more than one cycle of any length. Furthermore, by Lemma 2.15, every tree subgraph is thrackleable.

Now let *G* contain at most one odd cycle and no 4-cycles and let every connected component of *G* either contain exactly one cycle or be a tree. By Lemmas 2.13 and 11, the union of all connected components of *G* that include no odd cycles is thrackleable.

Consider a circle in the plane and place *v*₀ on the circle. Let *l* be the line that passes through *v*₀ and its antipodal point on the circle. Label one semi-circle bounded by *l* *L* and the other *R*. Arrange the connected components of *G* that contain no odd cycles so that for component *H*, one independent set of vertices lies in *L* and the other independent set of vertices is in *R*. Let *Cₘ* be an odd-cycle subgraph of *G*. Let the vertices of *Cₘ* be labeled *vᵢ*, *i ∈ Zₘ*, where *vᵢ* is adjacent to *vᵢ₊₁*. Arrange the non-zero vertices of *Cₘ* on the circle in *R* in the order described in Lemma 2.7 between *v*₀ and all other vertices in *R*. Arrange the odd vertices of *Cₘ* in the order described in Lemma 2.7 on the circle in *L* between *l* and the all the other vertices in *L* and on *l*. Draw the (*m − 1*)-path using straight-line segments from *v*₀ to *vₘ₋₁*. Because of the arrangement of vertices, the edges of this path cross all edges of all other components of *G*. Draw
\{v_{m-1}, v_0\} as a straight-line segment in the circle from \(v_{m-1}\) to a point between \(l\) and \(v_1\), then as an arc outside the circle to a point on the circle between \(v_{m-1}\) and \(v_{m-3}\). Cross only the edges between this point and \(v_0\) and terminate this edge at \(v_0\). This drawing is a thrackle. Lemma 1.8 concludes the proof.

In this section we have catalogued a number of types of thrackles. In some cases, the way those thrackle types relate to one another is clear. For example, since all even cycles are bipartite, all even cycles are shackleable. However, 6-cycles fit into more categories of thrackleable graphs than other even cycles. In Figure 15, we provide a pseudo-taxonomy of the thrackle types that we have discussed so far. We say "pseudo-taxonomy" because some graph types (in color) appear in more than one place.

![Figure 15: Partial Pseudo-taxonomy of Thrackles](image)

Through the proof of Woodall’s Theorem, we have developed tools to explore attempted proofs of the Thrackle Conjecture. Without the assumption that the Thrackle Conjecture is true, we no longer have an upper bound on the number of edges in a thrackle on \(n\) vertices. In the next section we explore efforts to find this upper bound.
3 Proving the Thrackle Conjecture in the Plane

Many mathematicians have tried to prove the Thrackle Conjecture in the plane. Woodall advanced the study of thrackles greatly, as we demonstrated in the previous section. Stephan Wehner (2010) has been cataloguing progress toward proofs of the Thrackle Conjecture on his website since 1996. In this section, we discuss the work of other mathematicians toward a proof of the Thrackle Conjecture, who attempt to suggest other compatible theorems and conjectures and find counterexamples to the Thrackle Conjecture.

3.1 Known Upper Bounds on Thrackles

A number of studies have used combinatorial approaches to find an upper bound for the number of edges on a thrackleable graph on \( n \) vertices. One clear upper bound comes from Theorem 1.2. Thus the greatest number of edges in any simple graph is bounded by the number of edges in a complete graph.

**Definition 3.1.** A complete graph on \( n \) vertices, \( K_n \), is a graph in which every pair of vertices is adjacent.

Since every vertex in \( K_n \) has degree \( n - 1 \), the upper bound on edges in any simple graph is \( \frac{n(n-1)}{2} = \binom{n}{2} \), by Theorem ???. We can reduce this bound further by considering the different characteristics of graphs. We generalize first about bipartite graphs, then about non-bipartite graphs.

3.1.1 Bipartite Thrackles

If a thrackleable graph is bipartite, it must also be planar as we show in Theorem 3.11, so Euler’s Formula reduces the upper bound on the number of edges in a thrackleable planar graph of order \( n \) to \( 3n - 6 \).
Theorem 3.2 (Euler’s Theorem, 1752). If $G$ is a connected, planar graph, then

$$|R(G)| = |E(G)| - |V(G)| + 2.$$

Corollary 3.3. If $G$ is a connected, planar graph with more than one edge, then

$$|E(G)| \leq 3|V(G)| - 6.$$

In Theorem 3.11 we show that thrackleable bipartite graphs are planar.

Lemma 3.4. Any thrackle of an $n$-cycle divides the plane into $\frac{n(n-3)}{2} + 2$ cells.

Proof. Note that $\frac{n(n-3)}{2} + 2 = 1 + \sum_{i=1}^{n-2} i$ for $n \geq 3$. We proceed by induction on $n$.

Consider a planar drawing of a 3-cycle. Since this drawing is a thrackle and the planar drawing of any cycle divides the plane into two cells, the claim holds. Now suppose that the claim holds for $n$, where $n \in \mathbb{N}_{\geq 4}$. Draw an $n$-cycle $C_n$ as a thrackle and number its consecutive vertices $v_0, v_1, \ldots, v_{n-1}$. We will show that the claim holds for $n + 2$. Note that $e_n = \{v_0, v_{n-1}\}$ crosses $n - 3$ other edges (it does not cross itself or the two edges adjacent to it). Delete $e_n$.

If $n$ is odd, arrange the vertices on a circle, as in Lemma 3.4. Insert $v_n$ between $v_{n-2}$ and $v_0$ on the circle; insert $v_{n+1}$ between $v_{n-1}$ and $v_1$ on the circle. If $n$ is even, complete an edge insertion as described in the proof of Lemma 2.13. Draw $e_n = \{v_{n-1}, v_n\}$. Notice that now $e_n$ crosses $n - 2$ edges (it now crosses $e_1$), but does not the number of cells in this drawing so far remains $\frac{n(n-3)}{2} + 2$. Draw edges $e_{n+1} = \{v_n, v_{n+1}\}$ and $e_{n+2} = \{v_{n+1}, v_{n+2}\}$, which increase the number of cells in the thrackle of this graph by $n$ and $n + 1$, respectively, through their crossings.
Thus, our drawing of the \((k + 1)\)-cycle contains
\[
\frac{n(n-3)}{2} + 2 + (n - 1) + n = 1 + \sum_{i=1}^{n-2} i + (n - 1) + n
\]
\[
= 1 + \sum_{i=1}^{n} i
\]
\[
= 1 + \sum_{i=1}^{(n+2)-2} i
\]
\[
= \frac{(n+2)((n+2) - 3)}{2} + 2
\]

Hence the claim holds. \(\square\)

We also rely on the following lemma, from Lovasz, Pach, and Szegedy. We require a definition for a cozy neighborhood to state and prove the lemma.

**Definition 3.5.** A cozy neighborhood of vertex \(v\) is an open set \(U\) containing \(v\) such that if a pair of edges crosses in \(U\), one must be incident to \(v\).

**Lemma 3.6** (Lovasz, Pach, and Szegedy, 1997). Let \(G\) be a thrackleable graph with cyclic subgraphs \(C_1\) and \(C_2\) that intersect at exactly one vertex, \(v\). Then \(C_1\) and \(C_2\) cross in a cozy neighborhood of \(v\) if and only if both cycles are odd.

**Proof following Lovasz, Pach, and Szegedy.** Suppose that thrackleable graph \(G\) contains two cycle subgraphs, \(C_1\) and \(C_2\), that intersect at exactly one vertex, \(v\). Let \(k_i\) be the length of \(C_i\), \(i = 1, 2\). If \(G\) is drawn as a thrackle, then the subthrackle of \(C_i\) divides the plane into \(\frac{k_i(k_i - 3)}{2}\) + 2 connected cells, by Lemma 3.4. Since the planar drawing of \(C_i\) divides the plane into two regions, cells in this subthrackle of \(C_i\) can be colored using just two colors, say black and white.

Consider the vertex \(v\). The two edges in \(C_2\) that are incident to \(v\) cross \(k_1 - 2\) edges in \(C_1\), while the remaining \(k_2 - 2\) edges in \(C_2\) cross \(k_1\) edges in \(C_1\). Thus \(C_2\) intersects \(C_1\) \(2(k_1 - 2) + k_1(k_2 - 2)\) times. This number is congruent to \(k_1k_2\) modulo 2. Let \(U\)
be a cozy neighborhood of \( v \). Consider the two edges of \( C_2 \) incident to \( v \). If \( C_2 \) crosses \( C_1 \) in \( U \), then the initial segment of one of these two edges is in a cell of one color and the other in a cell of a second color. Without loss of generality, let these cells be white and black, respectively. Yet this is possible if and only if \( k_1 k_2 \) is odd, in which case both cycles are odd.

\[ \text{Theorem 3.7 (Fulek and Pach, 2019). No thrackleable graph contains more than one 3-cycle.} \]

\[ \text{Proof.} \] Let \( G \) be a thrackleable graph. Then \( G \) contains no two disjoint 3-cycles, by Theorem 2.14. Moreover, \( G \) does not contain any pair of 3-cycles that share one or two edges, since it would then contain a 4-cycle. Thus, it suffices to consider the case in which two 3-cycles share exactly one vertex.

Toward a contradiction, suppose \( G \) contains a subgraph \( C \) that consists of two 3-cycle subgraphs, \( C_a \) and \( C_b \), which share one vertex, \( v_0 \). Let \( V(C_a) = \{v_0, v_1, v_2\} \) and \( V(C_b) = \{v_0, v_3, v_4\} \). Note that \( C_b \) divides the plane into two regions, say \( B \) and \( W \). Suppose that \( v_1 \in B \). Then as \( \{v_1, v_2\} \) crosses all three edges of \( C_a, v_2 \in W \). Yet for \( \{v_0, v_1\} \) and \( \{v_0, v_2\} \) to cross \( \{v_3, v_4\} \), either they must cross \( \{v_3, v_4\} \) more than once or \( \{v_1, v_2\} \) must cross \( \{v_3, v_4\} \) more than once.

\[ \text{Figure 16: Graph containing two 3-cycles that share a vertex} \]
Figure 17: Thrackle of a subgraph of previous figure

Hence no thrackleable graph contains more than one 3-cycle.

The following lemma about theta graphs allows us to prove the planarity of bipartite thrackleable graphs.

**Definition 3.8.** A theta graph is a graph composed of three internally disjoint paths that share endpoints. If, in a drawing of a theta graph, the clockwise order of its three paths is the same, then the drawing is a preserver; otherwise, it is a converter.

![Diagram of a theta graph](image)

Figure 18: Converter (left) and Preserver (right)

**Lemma 3.9** (Lovasz, Pach, and Szegedy, 1997). A theta-graph subdrawing of a thrackle is a converter if and only if at most one of its three paths has odd length.

**Proof.** Let $G$ be a thrackleable graph and $\Theta$ a theta subgraph of $G$, containing internally disjoint paths $P_1, P_2, P_3$. Let $u$ be the initial vertex of $P_1, P_2, P_3$ and $v$ their terminal vertex. Let $k_i$ be the length of $P_i$ for $i = 1, 2, 3$. Draw $G$ as a thrackle.
Let $U$ be a small neighborhood of $u$ and $V$ a small neighborhood of $v$. Without loss of generality, suppose that $P_2$ is oriented clockwise relative to $P_3$ in $U$. Since $P_2$ crosses $P_3$

$$2(k_3 - 1) + (k_2 - 2)k_3 \equiv k_2 k_3 \mod 2$$

(1)

times, $P_2$ is oriented counterclockwise relative to $P_3$ in $V$ only if both paths are odd.

Figure 19: Neighborhoods of Initial and Terminal Vertices in a Theta Graph

Now consider the $(k_2 + k_3)$-cycle $P_2 \cup P_3$. Color the cells created by the thrackle of $P_2 \cup P_3$ with black and white. Suppose that the initial edge of $P_1$ begins in a black cell. Because the path $P_1$ crosses $P_2 \cup P_3$

$$2[(k_2 - 1) + (k_3 - 1)] + (k_1 - 2)(k_2 + k_3) \equiv k_1 k_2 + k_1 k_3 \mod 2$$

(2)

times, the terminal edge of $P_1$ begins in a white cell only if $P_1$ is odd and exactly one of the other two paths is odd.

Suppose that $\Theta$ is drawn as a converter, with a clockwise orientation at $u$ and a counterclockwise orientation at $v$, without loss of generality. By hypothesis, if $P_2$ is clockwise relative to $P_3$ in both $U$ and $V$, then $P_1$ begins and terminates in the same color. Toward a contradiction, suppose that $P_2$ is clockwise relative to $P_3$ in $U$ and counterclockwise relative to $P_3$ in $V$. Then $P_1$ begins in black and ends in white or vice versa. Yet by equations (1) and (2), this situation is impossible. Therefore, at most one path is odd.
Now suppose that two paths in $\Theta$ are even. $P_2$ begins and ends clockwise relative to $P_3$, since $k_2k_3$ is even. Moreover, $P_1$ begins and ends in a black (white) cell, since $k_1k_2 + k_1k_3$ is even. Therefore, $\Theta$ is a converter.

To prove Theorem 3.11, we rely on Kuratowski’s Theorem and the four results following it.

**Theorem 3.10** (Kuratowski, 1930). A graph is planar if and only if it contains no subdivision of either $K_5$ or $K_{3,3}$.

We now have the tools to show that thrackleable bipartite graphs are planar.

**Theorem 3.11** (Lovasz, Pach, and Szegedy, 1997). If a bipartite graph is thrackleable, then it is planar.

*Proof following Lovasz, Pach, and Szegedy.* Let $G$ be a bipartite graph drawn as a thrackle. We will show by contradiction that $G$ contains subdivisions of neither $K_5$ nor $K_{3,3}$. Suppose that $G$ contains a subgraph $H$, which is a subdivision of $K_5$. Let the vertices of $H$ which correspond to $K_5$ be labeled $v_0, v_1, v_2, v_3, v_4$. Then two subcycles of $H$, one containing $v_0, v_1, v_2$ and one containing $v_0, v_3, v_4$, have exactly one vertex in common, $v_0$. Since $G$ is bipartite, both cycles are even. By Lemma 3.6, these two cycles do not cross in a cozy neighborhood of $v_0$. Yet this situation contradicts the assumption that $G$ is drawn as a thrackle. Therefore, $G$ does not contain a subdivision of $K_5$.

Now suppose that $G$ contains a subgraph $J$, which is a subdivision of $K_{3,3}$. Let the two classes of vertices be $L$ and $R$, with $u_1, u_2, u_3 \in L$ and $w_1, w_2, w_3 \in R$ corresponding to the vertices of $K_{3,3}$.

Suppose that all the paths in $J$ from vertices in $L$ to vertices in $R$ have the same parity. Delete the vertex $u_i$ and its incident edges to obtain a theta graph, $\Theta_i$, $i = 1, 2, 3$. If the paths’ parity is odd, then we contradict the membership of their endpoints in the same vertex class. Thus we consider the case in which the paths’ parity is even. By Lemma 3.9, $\Theta_i$ is a converter for all $i$, so $u_i$ and $u_j$ have different orientations for $i \neq j$, which is not possible.
Now suppose that at least two of the paths in $J$ have different parities. Then $\Theta_i$ contains an odd cycle for some $i$, which contradicts the assumption that $G$ is bipartite. Hence, by Kuratowski’s Theorem, $G$ is planar.

Now we have all the elements of a proof of an early upper bound on the number of edges in a bipartite thrackleable graph on $n$ vertices.

**Theorem 3.12** (Lovasz, Pach, and Szegedy, 1997). *A bipartite thrackleable graph on $n$ vertices contains at most $\frac{3}{2}n - 3$ edges.***

**Proof.** The proof follows from Theorem 3.11 Euler’s Formula, and Theorem 2.2. □

### 3.1.2 Non-bipartite Thrackles

Since we have an upper bound for thrackleable bipartite graphs, we now turn to thrackleable non-bipartite graphs. We begin with the following early theorem.

**Theorem 3.13** (Lovasz, Pach, and Szegedy, 1997). *Every thrackleable graph on $n$ vertices contains at most $2n - 3$ edges.*

Following Lovasz, Pach, and Szegedy, 1997. Let $G$ be a thrackleable graph on $n > 3$ vertices. By Theorem 3.12 if $G$ is bipartite, then $|E(G)| \leq \frac{3}{2}n - 3$. Therefore, it suffices to consider the non-bipartite case.

Let $C$ be the shortest odd cycle of $G$ and let $|V(C)| = c$. By the minimality of $c$ and Theorem 3.7 no vertex in $G \setminus C$ can be adjacent to two vertices in $C$. Moreover, by Theorem 2.14 $G \setminus C$ is bipartite.

Because $C$ is a cycle, cells of its thrackle can be colored black and white with no two cells that share an edge being the same color. Then in the thrackle of $G$, every vertex of $G \setminus C$ lies in either a black or white cell. Thus $|G \setminus C| = b + w$, where $b(w)$ is the number of vertices in black(white) cells. Furthermore, $b + w + c = n$. Without loss of generality, suppose that $b \leq w$. Then

$$b \leq \frac{n - c}{2}. \quad (3)$$

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Replace every vertex \( v \in C \) by \( v_b \) and \( v_w \), where \( v_b(v_w) \) is adjacent to every black (white) vertex formerly adjacent to \( v \). Suppose that \( v \) and \( v' \) are adjacent in \( C \). Insert edges \( \{v_b, v'_w\} \) and \( \{v_w, v'_b\} \), to obtain a doubling of \( C \) in \( G \).

![Diagram of Conway Doubling of an n-cycle](image)

Figure 20: The Conway Doubling of an \( n \)-cycle

By Theorem 2.11 this doubling of \( C \) is thrackleable. Denote by \( G' \) the augmentation of \( G \) containing the doubling of \( C \). Note that \( G' \) is bipartite and \( |E(G')| = |E(G)| + c \).

Thus by the previous theorem,

\[
|E(G')| - b = |E(G)| + c - b \leq \frac{3}{2}(n + c) - 3 - b \leq \frac{3}{2}(n + c) - 3,
\]

so

\[
|E(G)| + c \leq \frac{3}{2}(n + c) - 3 + b.
\]

By equation 3

\[
|E(G)| + c \leq 2n + c - 3.
\]

Therefore,

\[
|E(G)| \leq 2n - 3.
\]
Cairns and Nikolaevsky (2000) reduce this upper bound to $\frac{3}{2}(n - 1)$.

**Theorem 3.14** (Cairns and Nikolaevsky, 2000). Let $G$ be a thrackleable graph on $n$ vertices that contains an odd cycle $C_1$, which intersects every other cycle $C_2 \subseteq G$ an even number of times. $G$ contains at most $\frac{3}{2}(n - 1)$ edges.

**Proof.** Let $k_1$ denote the length of $C_1$. Perform a Conway doubling on $C_1$. Let $G'$ be the graph resulting from replacing $C_1$ with the doubled cycle in $G$. Since $G'$ is bipartite, we can use Euler’s Formula to solve for the number of edges in $G$.

$$2m' \geq 6(f - 1) + 2k_1$$  \hspace{1cm} (4)

Since $m' = m + k_1$ and $n' = n + k_1$,

$$m \leq \frac{3}{2}(n - 1).$$ \hspace{1cm} (5)

\[ \square \]

Fulek and Pach (2011) reduce this bound further to $|E(G)| \leq 1.428n$. They use a computational algorithm to reach this improvement. Xu (2014), finding this algorithm burdensome, uses basic facts about both graph theory and topology to achieve the bound $|E(G)| \leq 1.4(n - 1)$. This result is published in Goddyn and Xu (2017).

Fulek and Pach (2019) responded swiftly to Xu’s (2014) criticism by demonstrating that any thrackleable graph on $n$ vertices contains at most $1.3984n$ edges. Moreover, if a thrackleable graph contains no 3-cycle subgraphs, then it contains at most $1.3984(n - 1)$ edges. Most recently, Xu (2021) has reduced the known upper bound on the number of edges in a thrackleable graph on $n$ vertices to $1.393n$. Xu has done this by defining a new type of discharging rule.
As the progression of these upper bounds demonstrates, a combinatorial approach to proving the Thrackle Conjecture is difficult. Other attempts to prove the Thrackle Conjecture depend on proving that if there exists a thrackleable graph containing two cycles in the same connected component, then the Thrackle Conjecture is not true.
4 A Constructive Approach to the Thrackle Conjecture

In addition to combinatorial approaches to proving the Thrackle Conjecture, there are many graph theoretical and geometric ways to tackle the problem. We present in this section some innovative attempts to prove the Thrackle Conjecture.

4.1 Extension of Full Thrackles

Stephan Wehner (2010) conjectured that an counterexample to the Thrackle Conjecture must be in the form of a theta, dumb-bell, or figure-8 graph. Li, Daniels, and Rybnikov (2007) make this idea precise, by arguing that any counterexample should be constructible from an existing thrackle, particularly a full thrackle. For the remainder of this section, we consider only connected graphs.

Definition 4.1 (Li, Daniels, and Rybnikov, 2007). A full thrackle is a thrackle with the same number of edges and vertices.

For example, an n-cycle is a full thrackle on n vertices. Clearly, every vertex in an n-cycle is of degree 2. Other full thrackles exist on n vertices, as in Figure 21. In full thrackles that are not cycles there exist vertices of degree k, where k ≠ 2. Li, Daniels, and Rybnikov (2007) make this idea clear in Lemma 4.2

![Figure 21: Examples of Full Thrackleable Graphs](image)
**Lemma 4.2** (Li, Daniels, Rybnikov, 2007). If $G$ is a non-trivial graph that can be drawn as a full thrackle that is not a cycle, then $G$ contains at least one vertex of degree 1.

*Proof following Li, Daniels, Rybnikov, 2007.* Let $G$ be a non-cyclic thrackleable graph with $n$ vertices and edges. Toward a contradiction, suppose that $G$ contains no vertices of degree 1. Then all vertices of $G$ have degree 2 or higher. Since $G$ is not a cycle, at least one vertex must be of degree 3. Yet, this situation defies Theorem 1.2. Hence, $G$ must contain at least one vertex of degree 1.

To classify thrackleable graphs, Li, Daniels, and Rybnikov (2007) introduce the idea of an incidence type.

**Definition 4.3.** The incidence type of a thrackleable graph $G$ is a non-decreasing degree sequence for $G$.

One incidence type, described in Lemma 4.4, impossible for full thrackles.

**Lemma 4.4** (Li, Daniels, Rybnikov, 2007). Let $G$ be a full thrackleable graph on $n$ vertices. Then $G$ cannot have the incidence type $(1,1,\ldots,1,j,k)$, where $2 \leq j \leq k$.

*Proof.* Toward a contradiction, suppose that $G$ is a full thrackleable graph with the incidence type $(1,1,\ldots,1,j,k)$. Call the vertex with degree $j(k)$ $v_j(v_k)$. Since $n-2$ vertices have degree 1, $j+k = n+2$, by Theorem 1.2. Yet both $v_j$ and $v_k$ can be adjacent to at most $n-1$ other vertices, so $G$ cannot be full thrackleable.

If we imagine that there exists a counterexample to the Thrackle Conjecture, we must be able to construct it from a full thrackle, by Lemma 1.7. To disprove the Thrackle Conjecture, it would suffice to show that some full thrackleable graph is extensible.

**Definition 4.5.** A thrackleable graph $G$ is extensible if inserting a new edge between two existing vertices in $V(G)$ results in a thrackleable graph.

Figure 22 is an example of a thrackleable graph that is extensible.
Before proving the next theorem, we need the following lemma. In fact, this result is implied by Woodall’s Theorem. We provide a proof nonetheless.

**Lemma 4.6** (Woodall, 1971). *Every connected full thrackleable graph contains exactly one cycle subgraph.*

*Proof.* Let $G$ be a connected, full thrackleable graph on $n$ vertices. Since $G$ contains as many edges as vertices, it is not a tree. Suppose that $G$ contains two cycle subgraphs, $C_1$ and $C_2$ on $l$ and $m$ vertices, respectively. Let $k$ be the number of edges that $C_1$ and $C_2$ share and let $j$ be the length of the path between $C_1$ and $C_2$. Then the minimal connected subgraph of $G$ containing both $C_1$ and $C_2$, say $H$, contains $l + m - k + j$ edges.

Since $H$ contains only two cycle subgraphs, at most one of $j$ and $k$ can be positive. If $k$ is positive, then $C_1$ and $C_2$ share at least $k + 1$ vertices, so $H$ contains at most $l + m - (k + 1) + j$ vertices and $H$ is not full thrackleable. If $j$ is positive, then $C_1$ and $C_2$ each contain one vertex of the $J$-path, so $H$ contains $l + m - k + (j + 1)$ vertices, so $H$ is not full thrackleable. If both $j$ and $k$ are zero, then $C_1$ and $C_2$ share a vertex and $H$ contains $l + m - 1$ vertices and $l + m$ edges and is not full thrackleable. Since $G$ is connected, it is formed by adding at least as many vertices as edges to $H$. Therefore, $G$ cannot contain more than one cycle subgraph. \qed

We show now in Theorem 4.7 that if a full thrackleable graph can be extended, the resulting graph must contain two cycle subgraphs.
Theorem 4.7 (Wehner, 2010). If $G$ is the extension of a full thrackleable graph, then $G$ contains either a theta graph, a figure-8, or a dumbbell, and no other cycle subgraphs.

Proof. Suppose that $H$ is a full thrackleable, extensible graph. By Lemma 4.6, $H$ contains exactly one cycle subgraph, say $C$. Let $G$ be an extension of $H$, with $G = H \cup \{u, w\}$ where $u, w \in H$. Let $P_u$ ($P_w$) be the shortest path in $H$ beginning at $u$ ($w$) and terminating at $C$. Denote by $v_u$ ($v_w$) the terminal vertex of $P_u$ ($P_w$), as in Figure 23.

Suppose that $P_u$ and $P_w$ intersect at some vertex or vertices not in $C$. Starting at $u$, let $z$ be the first such vertex where the two paths meet and let $Q$ be the shortest path from $z$ to $C$. Then $G$ contains a dumbbell, $J$ (shown in Figure 24), consisting of $C$, the cycle subgraph containing $u, w$, and $z$, and the path $Q$.

Now suppose that $P_u$ and $P_w$ are disjoint outside $C$, but intersect at some $x \in C$. Then $G$ contains a figure-8 subgraph, $K$, as in Figure 25, whose two constituent cycles are $C$ and the cycle containing $u, w$, and $x$. Note that in this case $x = v_u = v_w$. 
Finally, suppose that $P_u$ and $P_w$ are disjoint. Then in $G$, $v_u$ and $v_w$ are the initial and terminal vertices, respectively, of three paths forming a theta graph, $\Theta$, as in Figure 26. Note that this case allows for the possibility that $P_u$ ($P_w$) has length zero and $v_u = u$ ($v_w = w$).

Recall that in each case, $G \setminus J$, $G \setminus K$, $G \setminus \Theta$, is a subgraph of $H$ consisting of one or more trees. This concludes the proof.

Although we do not yet know whether any full thrackleable graph is extensible, there is evidence that some $n$-cycles are not extensible.

**Lemma 4.8** (Li, Daniels, Rybnikov, 2007). *The 3-cycle (with incidence type $(2,2,2)$) is not extensible.*

**Proof.** Since the 3-cycle is already a complete graph on three vertices and thrackles are only defined on simple graphs, no additional edges are possible on the three given vertices.
Although Li, Daniels, and Rybnikov (2007) demonstrated the above result for 3-cycles, we find that it is also true for 5-cycles.

**Lemma 4.9.** The 5-cycle (with incidence type $(2, 2, 2, 2)$) is not extensible.

*Proof.* Consider the 5-cycle. Any new edge drawn using existing vertices creates a forbidden configuration: either a double edge or a 4-cycle subgraph. □

Li, Daniels, and Rybnikov (2007) make Wehner’s hypothesis precise by describing the 1-2-3 set, shown in Figure 27.

**Definition 4.10** (Li, Daniels, Rybnikov, 2007). A **1-2-3 set** is a full thrackleable graph on $n$ vertices with incidence type $(1, 2, \ldots, 2, 3)$, where $n - 2$ vertices are of degree 2.

![Figure 27: A 1-2-3 Set](image)

**Lemma 4.11** (Li, Daniels, Rybnikov, 2007). A 1-2-3 set contains a body consisting of a cycle and a tail that is a path, the initial vertex of which is in the body.

*Proof.* Let us proceed by induction. Consider the simplest 1-2-3 set, with incidence type $(1, 2, 2, 3)$. In this graph, the body is a 3-cycle and the tail is a 1-path that terminates at one vertex of the cycle.

![Figure 28: The Simplest 1-2-3 Set](image)
Now suppose that $G$ is a 1-2-3 set on $n$ vertices, that consists of a body and a tail, which is a path between the vertices of degree 1 and 3. We will insert a vertex and edge to obtain a 1-2-3 set on $n + 1$ vertices. We consider two cases: first, a body with at least 5 vertices; then, a body with 3 vertices.

Suppose that the body of $G$ contains at least 5 vertices, as in Figure 29. If we attach a leaf to any vertex in the cycle, then the result is not a 1-2-3 set. If we attach a leaf to the terminal vertex of the tail, then the former degree-1 vertex becomes a degree-2 vertex, while the inserted vertex is of degree 1. Thus, the resulting graph contains $n + 1$ vertices and has incidence type $(1, 2, \ldots, 2, 3)$, with $n - 1$ vertices of degree 2. Alternatively, we can insert a vertex and edge in the cycle or between interior vertices of the tail to obtain the same incidence type.

![Figure 29: Potential Edge Insertions in a 1-2-3 Set Containing an $n$-cycle ($n \geq 5$)](image)

Now suppose that the body of $G$ contains only 3 vertices, as in Figure 30. Then we can use the preceding argument, except for the insertion of one edge and one vertex in the body—this results in a 4-cycle subgraph. Instead, we can insert two vertices and two edges in the body to obtain a 1-2-3 set on $n + 2$ vertices.

![Figure 30: Potential Edge Insertions in a 1-2-3 Set Containing an 3-cycle](image)
If it can be proved that 1-2-3 sets cannot be extended, then the Thrackle Conjecture is true. Li, Daniels, and Rybnikov conclude their discussion of 1-2-3 sets by presenting the following conjecture.

**Conjecture 4.12** (Li, Daniels, Rybnikov, 2007). *No 1-2-3 set is extensible.*

This conjecture implies the following theorem immediately.

**Theorem 4.13** (Li, Daniels, Rybnikov, 2007). *If the 1-2-3 Conjecture is true, then no full thrackle is extensible.*

**Proof.** Suppose the 1-2-3 Conjecture is true. Let $H$ be a full thrackleable graph. Toward a contradiction, suppose that $H$ is extensible to $G$. By Theorem 4.7, $G$ contains a subgraph that is either a theta graph, a dumbbell, or figure-8. Denote this subgraph $J$. Consider a vertex $u \in J$ of degree $k > 2$. Let $e$ be an edge in one of the cycle subgraphs of $J$ so that $e$ is incident to $u$. Removing $e$ results in a 1-2-3 set. Thus $J$ is an extension of a 1-2-3 set, a contradiction.

As the preceding theorem shows, the Thrackle Conjecture can be reduced to proving the 1-2-3 Conjecture. In the next section we look at cases where the Thrackle Conjecture is true.

### 4.2 Clues about the Thrackle Conjecture from Design Theory

For a number of types of graphs the Thrackle Conjecture clearly holds. In particular, the Thrackle Conjecture holds for straight-line thrackles, as we prove in Theorem 4.22. Recall that a straight-line thrackle is a thrackle of a graph in which every edge is represented by a straight-line segment. Fisher’s Inequality and a combinatorial result from Paul Erdős can be applied to the Thrackle Conjecture to show us that it holds for straight-line thrackles. In particular, the Thrackle Conjecture holds for *planar thrackles*, a fact given by Ryser (1968).
4.2.1 Planar Thrackles

The class of graphs whose thrackleability is given by Fisher’s Inequality is described by the term planar thrackles. This term is used in multiple ways in the literature, but for the purpose of this paper, we use the following definition.

**Definition 4.14.** A **planar thrackle** is a thrackle of a graph in which no edges cross.

One such graph type is a **star**.

**Definition 4.15.** A **star** is a tree on $n$ vertices, in which one central vertex has degree $n - 1$ and $n - 1$ vertices have degree 1.

![Figure 31: Star on Five Vertices](image)

We demonstrate below how Fisher’s Inequality relates to the Thrackle Conjecture. We need the concept of an **incidence matrix** for our explanation.

**Definition 4.16.** Let $S$ be a finite set and $\mathcal{F}$ a collection of subsets of $S$, with $|S| = m$ and $|\mathcal{F}| = n$. For $S_j \subseteq S$, set $a_{ij} = 1$ if $a_i \in S_j$ and $a_{ij} = 0$ otherwise. The resulting $m \times n$ matrix $A = (a_{ij})$ is called an **incidence matrix**.

![Figure 32: $C_3$ with Incidence Matrix](image)
Theorem 4.17 (Fisher’s Inequality, 1938). Let $S$ be a finite set and let $\mathcal{F}$ be a collection of subsets of $S$ such that $|S_i \cap S_j| = 1$ for $i \neq j$. Then $|\mathcal{F}| \leq |S|$.

Proof. Let $A$ be the incidence matrix for thrackleable graph $G$. Then $A^T A$ is a square, non-singular matrix. Toward a contradiction, suppose that $m < n$. Then we can create an $n \times n$ square matrix $A_0$ from $A$ by inserting $n - m$ rows of zero entries below the last row of $A$. Then $A_0^T A_0 = A^T A$, but $A_0^T A_0$ is singular. Thus the number of edges of $G$ is bounded above by the number of its vertices. \(\square\)

Fisher’s Inequality tells us that the number of elements in $\mathcal{F}$ is bounded above by the number of elements in $S$. Relating to graphs, this means that if every pair of edges in graph $G$ is adjacent, then the number of edges of $G$ is bounded above by the number of its vertices. Thus Fisher’s Inequality gives us that the 3-cycle, 1- and 2-path, and all stars obey the Thrackle Conjecture. These are exactly the class of planar thrackles.

Yet the Thrackle Conjecture holds for a larger class of graphs than just stars and the 3-cycle. It holds for all straight-line thrackles. Let us demonstrate which graphs can be drawn as straight-line thrackles. We mention pointed vertices, caterpillars, and spiders in the following theorem and its proof, so we must first define them.

Definition 4.18. A caterpillar is a tree in which every vertex is adjacent to at most two interior vertices.

![Caterpillar](image33)

Figure 33: Caterpillar
Definition 4.19. A spider is a tree in which exactly one vertex, $v_0$, has degree at least 3 and all others have degree at most 2. The degree of $v_0$ is the number of legs of the spider. The length of each leg is the length of the path beginning at a degree-1 vertex and terminating at $v_0$.

![Spider with Three Legs of Length 2](image)

Definition 4.20. A pointed vertex is a vertex, $v$, whose incident edges all lie in a half-plane bounded by a line through $v$.

![Pointed Vertex](image)

We use all these terms in either the statement or proof of the following theorem.

Theorem 4.21 (Woodall, 1971). A finite graph $G$ is straight-line thracklable if and only if either

1. $G$ contains an odd cycle and every vertex of $G$ is adjacent to a vertex of the cycle or

2. $G$ is a disjoint union of caterpillars
Proof. We streamline the proof presented by Woodall.

In Lemma 2.7 we demonstrated that odd cycles can be drawn as straight-line musquashes. Now we show that if $G$ consists of an odd cycle with leaves attached only to vertices of the odd cycle, then $G$ is straight-line thrackleable.

Let $C_n$ be an odd cycle drawn as a straight-line musquash and consider $v \in C_n$. Since $v$ is pointed through circular symmetry, the two edges incident to it, $e$ and $e'$, form an acute angle, say $e$ is at angle $\theta$ from $e'$. Note the region bounded by $v, e, e'$, and the boundary of the circle underlying the musquash. Edges adjacent to either $e$ or $e'$ have one vertex on the boundary of this region and one outside this region, whereas edges that are not adjacent to either cross from outside the region, to inside, to outside again.

Let $f_i$ denote a 1-path ending in a leaf on the circle underlying the musquash of $C_n$. Now let $\phi_i$ be the angle between $f_i$ and $e$, where $0 < \phi_i < \theta$ and $\phi_i \neq \phi_j$ when $i \neq j$. Attach any number of leaves to $v$ with these characteristics. No $f_i$ will cross $e, e'$, or $f_j$ for $i \neq j$ because of the size of $\phi_i$, but they will all cross the edges of $C_n$ that are crossed by both $e$ and $e'$ or adjacent to other vertices of $C_n$. Moreover, all of the $f_i$ cross from one side of the underlying circle to the other. Thus attaching straight-line segments to any vertex of $C_n$ produces a straight-line thrackle.

Let $G$ be a graph constructed by attaching leaves to vertices of an odd cycle, as described above. By the construction of this musquash, the even-indexed vertices of the cycle are grouped together on the underlying circle, as are the odd-indexed vertices. Let $W_L$ be a neighborhood of the even vertices of the cycle and the terminal vertices of leaves emanating from odd vertices. Similarly, let $W_R$ be a neighborhood of the odd vertices of the cycle and terminal vertices of leaves emanating from the even vertices. Notice that removing any edge from the odd cycle in $G$ produces a caterpillar, which is a shackle in this drawing. Moreover, every caterpillar can be constructed this way, so by Lemma 1.7 every caterpillar can be drawn as a straight-line thrackle. This demonstrates that a disjoint union of caterpillars is straight-line thrackleable.
Now we show that if $G$ is straight-line thrackleable, then it must either contain an odd cycle with all vertices of $G$ adjacent to a vertex of the cycle or $G$ must be a finite union of caterpillars. It suffices to show that even cycles and spiders on three legs of length two are not straight-line thrackleable.

Suppose, toward a contradiction, that an even cycle is straight-line thrackleable. Consider $v, e,$ and $e' \in C_n$ and the region they bound as described above and depicted in Figure 36. All vertices that are not incident to either $e$ or $e'$ must be on one side of this region or the other. Moreover, every remaining straight-line segment edge must pass through this region. Yet since only an odd number of vertices remain, there must be one edge that does not cross this region. Thus we do not have a thrackle of $C_n$.

![Figure 36: Even Cycle](image)

Now consider a tree that is not a caterpillar. Any such tree has a spider on three legs of length 2, as in Figure 37 as a subgraph. Fix an orientation about the central vertex, $v_0$. For $\{v_1, v_2\}$ to cross $\{v_0, v_3\}$ and $\{v_0, v_5\}$ as a straight-line segment, the angle from $\{v_0, v_1\}$ to $\{v_0, v_5\}$ passing through $\{v_0, v_3\}$ must be acute. This configuration also allows $\{v_5, v_6\}$ to cross $\{v_0, v_3\}$ and $\{v_0, v_1\}$ as a straight-line segment. However, it is not possible for $\{v_3, v_4\}$ to cross $\{v_0, v_1\}$ and $\{v_0, v_5\}$ as a straight-line segment, because the angle from $\{v_0, v_1\}$ to $\{v_0, v_5\}$ is obtuse.
Figure 37: Spider with Three Legs of Length 2

Now that we know which graphs can be drawn as straight-line thrackles, we show that the Thrackle Conjecture holds for this entire class of graphs.

**Theorem 4.22** (Erdos, 1946). *The Thrackle Conjecture holds for straight-line thrackles.*

**Proof.** Following Perles, in Pach and Sterling (2011)

Let $G$ be drawn as a straight-line thrackle. For every pointed vertex in $G$, we delete the right-most edge incident to it. Toward a contradiction, suppose that $e_k = \{v_j, v_k\}$ remains after these deletions. Then $e_k$ is not the right-most edge relative to either $v_j$ or $v_k$. Therefore, there must exist $v_i, v_l \in V(G)$ with $\{v_i, v_j\}$ and $\{v_k, v_l\}$ lying on the same line as $e_k$. Since $\{v_i, v_j\}$ and $\{v_k, v_l\}$ are not adjacent and do not intersect, $G$ is not drawn as a thrackle.

For the purpose of organization, let us give a name to the class of thrackleable graphs that contain an $n$-cycle: *fuzzy cycles*. They are "fuzzy" because if one edge were removed from the cycle subgraph, the remaining graph would be a caterpillar. As described before, only fuzzy odd cycles are straight-line thrackleable. In Figure 38 we present a taxonomy of the graphs discussed in this section.
The discussion in this paper so far demonstrates the limits of knowledge about the Thrackle Conjecture in the plane. In the next section we look at the Thrackle Conjecture on other surfaces and what those results might mean for thrackles in the plane.
5 Thrackles on Surfaces

As the discussion so far demonstrates, the study of thrackles in the plane has not shown whether the Thrackle Conjecture is true. Here we look to other surfaces for clues about thrackles.

**Definition 5.1.** An \( m \)-manifold is a Hausdorff, second-countable space in which every point is contained in a neighborhood homeomorphic to \( \mathbb{R}^m \). A 2-dimensional manifold is called a surface.

By the definition of a manifold, every graph that is thrackleable in the plane is thrackleable on any surface. However, certain graphs that are thrackleable on some surfaces are not thrackleable in the plane. In this section we explore the thrackleability of graphs on the sphere, torus, and other surfaces.

### 5.1 Great Circle Thrackles

Using a composition of a homeomorphism between an open subset of the sphere and an open subset of \( \mathbb{R}^2 \), and the drawing of a thrackle in the plane, we can draw any plane thrackle on the sphere. Moreover, we know that we can draw thrackles on the sphere with at most as many edges as vertices using great circles (Cairns, Koussas, and Nikolayevsky, 2015). In this section we prove that the Thrackle Conjecture is true for standard great circle thrackles.

**Definition 5.2.** A great circle of a sphere is a circle whose diameter equals the diameter of the sphere, and whose center is the center of the sphere (Figure 39).
Note that great circles on the sphere are analogous to straight lines in the plane. Although there are infinitely many great circles between any pair of antipodal points, there is a unique great circle between any two non-antipodal points on the sphere. We can draw many thrackleable graphs on the sphere as great circle thrackles.

**Definition 5.3** (Cairns, Koussas, and Nikolayevsky, 2015). A great circle thrackle is a thrackle drawn on the sphere, whose vertices are represented as points on the sphere and whose edges are represented by the arcs of great circles as in Figure 40.
We represent the great circle containing edge $e$ as $C(e)$ and denote the positive and negative hemispheres of $C(e)$ by $H_e^+$ and $H_e^-$, respectively. If $e$ is shorter than $\pi$ we call it a \textit{short edge} and if $e$ is longer than $\pi$ we call it a \textit{long edge}. In addition to the long/short classification, every edge in a great circle thrackle is assigned a direction.

Despite there being multiple ways to represent a thrackle on the sphere, we consider what Cairns, Koussas, and Nikolayevsky (2015) term a \textit{standard great circle thrackle} in \textit{general position}. Figure 41 provides an example of a standard great circle thrackle and a counterexample.

\textbf{Definition 5.4.} A \textit{great circle thrackle drawn in general position} is a thrackle in which no three vertices are drawn in the same great circle and no two vertices are represented by antipodal points. A \textit{standard great circle thrackle} is a great circle thrackle of a connected graph without any leaves, drawn in general position, as in Figure 41.
We characterize cycle and path subgraphs of thrackleable graphs as either good or bad, depending on the crossing orientation of their edges. In addition to the direction assigned to each edge, the great circle containing each edge bounds hemispheres assigned either a positive or negative value.

**Definition 5.5.** Let $G$ be a graph containing at least two edges, drawn as a standard great circle thrackle. Consider edges $e, f, g \in G$. If $f$ is directed toward the negative hemisphere bounded by $\mathcal{C}(e)$ and $g$ toward the positive hemisphere, then the crossing orientation of $e$ with respect to $f$, $\chi(e, f)$, equals $-1$ and the crossing orientation of $e$ with respect to $g$, $\chi(e, g)$, equals $1$ (Figure 42).
Because crossing orientation is antisymmetric, then $\chi(f, e) = -1$ and $\chi(g, e) = 1$.

A directed $k$-walk is **good** if $\chi(e_i, e_{i+1}) = 1$ for all $i \in \mathbb{Z}_k$ or $\chi(e_i, e_{i+1}) = -1$ for all $i \in \mathbb{Z}_k$. Any directed $k$-walk that is not good is **bad**. These conditions also describe good and bad directed $k$-paths. Similarly, a good $k$-cycle is any directed $k$-cycle in which every directed subpath is **good**; any other directed $k$-cycle great circle drawing is **bad**.

We demonstrate good and bad $k$-paths and $k$-cycles in Figures 43 and 44.
5.1.1 Introductory Lemmas about Great Circle Thrackles

Now that we have the terminology necessary to describe great circle thrackles, we require several lemmas characterizing great circle thrackles to prove that the Thrackle Conjecture holds. We begin with simple statements about paths and the 3-cycle.

**Lemma 5.6** (Cairns, Koussas, and Nikolayevsky, 2015). *In any standard great circle thrackle, no two long edges are adjacent.*

*Proof following Cairns, Koussas, and Nikolayevsky (2015).* Let $T$ be the thrackle of some graph $G$. Toward a contradiction, suppose that $e$ and $f$ are adjacent long edges drawn in $T$. Let $v$ be the vertex incident to both $e$ and $f$. Since every great circle containing the point corresponding to $v$ must also contain its antipodal point, $v'$, and both $e$ and $f$ are long, $e$ and $f$ must cross at $v'$, which contradicts $T$ being a thrackle. 

We further characterize standard great circle drawings of $n$-cycles in the following two lemmas.
Lemma 5.7 (Cairns, Koussas, and Nikolayevsky, 2015). In every bad drawing of a walk containing exactly three distinct edges, at least one edge is long and the center edge is short.

Proof. Let $G$ be a great circle thrackleable graph and let $W = e_1e_2e_3$ be a bad walk in $G$, where $e_1, e_2, e_3$ are distinct. Without loss of generality, let $\chi(e_1, e_2) = 1$. Toward a contradiction, suppose that $e_2$ is drawn long, as in Figure 45. Then by Lemma 5.6, $e_1$ and $e_3$ are drawn short. Moreover, because $e_2$ is long, $v_2$ is in $H_{e_1}^-$. Since $e_3$ must cross $e_1$, and $v_1$ is in $H_{e_2}^+$, $\chi(e_2, e_3) = 1$, meaning $W$ is good. Therefore, if $W$ is bad, the center edge must be short.

Figure 45: Bad 3-Walk Containing a Long Center Edge

Now, since $\chi(e_1, e_2) = 1$ by hypothesis, it must be true that $\chi(e_2, e_3) = -1$. Since $v_1$ and $v_2$ bound $e_2$, both $v_1$ and $v_2$ lie on $C(e_2)$. By convention, $v_0$ and $v_3$ each lie in some hemisphere of $C(e_2)$.

We assume toward a contradiction that $e_1$ and $e_3$ are drawn as short edges. Then $v_0$ and $v_3$ lie in different hemispheres, so $e_1$ and $e_3$ do not cross and $G$ is not drawn as a thrackle. Thus, either $e_1$ or $e_3$ must be drawn long. Figure 46 demonstrates this necessity.

\qed
Remark. It is, perhaps, necessary to highlight the fact that all standard great circle thrackles are directed. In the final paragraph of the preceding proof, the directedness of the walk is essential. If we do not assume directedness, then forcing both $e_1$ and $e_3$ to be short does not require $v_0$ and $v_3$ to be in different hemispheres of $E(e_2)$. If the assumption of directedness were removed, then the drawing in Figure 47 would be possible. Therefore, the contradiction we reach in the above proof would not exist.
Lemma 5.8 (Cairns, Koussas, and Nikolayevsky, 2015). In every bad drawing of a 3-path in a standard great circle thrackle, the center edge is an edge in a bad 3-cycle.

Proof. Let $P = e_1e_2e_3$ be a bad 3-path. By Lemma 5.7, $e_2$ is short and either $e_1$ or $e_3$ is long. Without loss of generality, let $e_1$ be long. Because $P$ is drawn as a standard great circle thrackle, more than one edge must be incident to $v_0$. Let $f$ be incident to $v_0$. Because $v_0$ is situated between $e_2$ and $e_3$ and $f$ is short and must meet both $e_2$ and $e_3$, $f$ must be incident to $v_2$. Therefore, $e_1e_2f$ is a bad 3-cycle.

\[\square\]

5.1.2 $N$-Cycle Great Circle Thrackles

As in the plane, every $n$-cycle has a standard great circle thrackle.

Theorem 5.9. There exists a standard great circle thrackle of every $n$-cycle for $n \geq 3$, $n \neq 4$.

Proof. As the shortest distance between any two points on the sphere is determined by the great circle containing them, every straight-line thrackleable graph in the plane is great circle thrackleable on the sphere. Therefore, every odd cycle is great circle thrackleable on the sphere, by Lemma 2.7. See two examples in Figure 48.
Now we turn to even cycles. As in the plane, the 4-cycle is not great-circle thrackleable on the sphere. Let $C$ be a 4-cycle. Toward a contradiction, suppose it is possible to draw $C$ as a great circle thrackle. Consider $C(e_1)$. Suppose that both $e_2$ and $e_4$ are both long or both short, as in Figure 49 (left). Since $e_2$ and $e_4$ must cross each other, their endpoints ($v_2$ and $v_3$, respectively) are in the same hemisphere of $C(e_1)$, say $H_{e_1}^+$. Then the two endpoints of $e_3$ are in the same hemisphere, so $e_3$ either is short and does not cross $e_1$ or $e_3$ is long and crosses $e_1$ twice, so $C$ is not drawn as a thrackle. Alternatively, suppose that $e_2$ is short and $e_4$ is long, as in Figure 49 (right). Then $v_2$ is in $H_{e_1}^+$ and $v_3$ is in $H_{e_1}^-$. Yet in this scenario $e_3$ must cross $e_4$, so again $C$ is not drawn as a thrackle. Therefore, the 4-cycle is not great circle thrackleable.
However, every even $n$-cycle for $n \geq 6$ is great circle thrackleable. One great circle thrackle of the 6-cycle is formed by alternating long and short edges. Then given any great circle thrackle of an even $n$-cycle, we form a great circle thrackle of an $(n+2)$-cycle by replacing an edge with three edges, as in the proof of Lemma 2.13. We demonstrate this in Figure 50.
We now prove that for any odd cycle of length greater than three, no great circle thrackle contains any long edges. First, we need the following series of lemmas. In particular we need to know how to calculate the crossing orientations of edges within great circle thrackles. Lemma 5.10 lays the foundation for those calculations.

**Lemma 5.10** (Cairns, Koussas, and Nikolayevsky, 2015). *In a standard great circle thrackle of graph $G$, every edge adjacent to long edge $e$ lies in the same hemisphere of $C(e)$.***

**Proof following Cairns, Koussas, and Nikolayevsky (2015).** Let $G$ be drawn as a great-circle thrackle and let $feg$ be a 3-path in $G$. Toward a contradiction, suppose that $f$ and $g$ lie in different hemispheres of $C(e)$. By Lemma 5.6, both $f$ and $g$ are short, so they do not cross and are not adjacent, which contradicts $G$ being drawn as a thrackle.

Given the result above, we can use the following lemma to calculate the crossing orientations of short edges adjacent to long edges.

**Lemma 5.11** (Cairns, Koussas, and Nikolayevsky, 2015). *If $e_1e_2e_3$ is a directed 3-path in which $e_2$ is long, then $\chi(e_1,e_2) = \chi(e_1,e_3)$.***

**Proof following Cairns, Koussas, and Nikolayevsky, 2015.** By Lemma 5.6, both $e_1$ and $e_3$ are short. Moreover, by Lemma 5.10 $e_1$ and $e_3$ lie in the same hemisphere of $C(e_2)$. Without loss of generality, let $\chi(e_1,e_2) = 1$. Then in a small neighborhood of $v_1$, $e_2$ lies in $H(e_1)^+$. Yet $v_2$ lies in $H(e_1)^-$ because $e_2$ is long. Because $e_3$ is short and must cross $e_1$, $v_3$ lies in $H(e_2)^+$. As $e_3$ is directed toward $v_3$, $\chi(e_1,e_3) = 1$.

The following lemma allows us to calculate the crossing orientations of paths of short edges.

**Lemma 5.12** (Cairns, Koussas, and Nikolayevsky, 2015). *Let $P = e_1e_2\ldots e_m$ be a directed path consisting only of short edges. Let $e$ be an edge that is not incident to any internal vertex of $P$. Then $\chi(e,e_i) = (-1)^{i-1}\chi(e,e_1)$ for $1 \leq i \leq m$.***
Proof. Without loss of generality, let $\chi(e, e_1) = 1$. Suppose that $m = 2$. Then $v_1$ lies in $\mathcal{H}(e)^+$. Since $e_2$ is short, $v_2$ lies in $\mathcal{H}(e)$, so $\chi(e, e_2) = -1$. Now suppose that $\chi(e, e_{m-1}) = (-1)^{m-2}\chi(e, e_1)$. Because $e_m$ is short, $v_m$ is in the opposite hemisphere of $\mathcal{H}(e)$ to $v_{m-1}$. Moreover, $e_m$ is directed toward $v_m$ and away from $v_{m-1}$. Hence, $\chi(e, e_m) = -\chi(e, e_{m-1}) = (-1)^{m-1}\chi(e, e_1)$.

In Lemma 5.13 we further characterize paths.

**Lemma 5.13** (Cairns, Koussas, and Nikolayevsky, 2015). In every good, directed path, consecutive long edges are separated by an odd number of short edges.

Proof. Suppose that $e_1 e_2 \ldots e_m$ is a good directed path, where $e_1$ and $e_m$ are long. Without loss of generality, let $\chi(e_i, e_{i+1}) = 1$ for $1 \leq i < m$. Let $x = e_1 \cap e_m$, $y = e_2 \cap e_m$, and $z = e_1 \cap e_{m-1}$, as in Figure 51. Let $A$ be the 3-cycle formed by $a_1 = \{v_1, x\}$, $a_2 = \{v_1, y\}$, and $a_3 = \{x, y\}$, bounding the orange region in Figure 51. Let $B$ be the 3-cycle formed by $b_1 = \{v_3, x\}$, $b_2 = \{v_3, z\}$, and $b_3 = \{x, z\}$ bounding the pink region in Figure 51.

Toward a contradiction, suppose that $m$ is even. Then by Lemmas 5.11 and 5.12, $\chi(a_1, a_2) = \chi(e_2, e_m) = -\chi(e_1, e_2) = -\chi(a_2, a_3)$ and $\chi(b_1, b_2) = \chi(e_4, e_3) = -\chi(e_{m-1}, e_1) = -\chi(b_2, b_3)$. Thus by Lemma 5.7, $A$ and $B$ are bad 3-cycles and each must contain a long edge. As $a_2 \subseteq e_2$ and $b_2 \subseteq e_{m-1}$, neither of these segments can be long. In $A$, either $a_1$ or $a_3$ is long and in $B$, either $b_1$ or $b_3$ is long. Notice that all four of these segments contain $x$ as an endpoint. Thus the long segments in both $A$ and $B$ contain both $x$ and its antipodal point. Therefore, they meet twice, which contradicts the assumption of a thrackle drawing. Therefore, $m$ must be odd.
The following lemma demonstrates that for \( n > 3 \), no bad drawing of an \( n \)-cycle is possible.

**Lemma 5.14** (Cairns, Koussas, and Nikolayevsky, 2015). *Every standard great circle thrackle of an \( n \)-cycle is good for \( n \geq 5 \).*

*Proof.* If there existed a bad standard great circle thrackle of \( n \)-cycle \( C \) (\( n \geq 5 \)), then it would contain a bad 3-path and must contain a bad 3-cycle, by Lemma 5.8. This 3-cycle subgraph of an \( n \)-cycle is impossible for \( n \geq 5 \). Therefore, any \( n \)-cycle fulfilling the hypotheses must be good. \( \square \)
Before proving the main result, we characterize good paths and good cycles.

**Lemma 5.15** (Cairns, Koussas, and Nikolayevsky, 2015). Let $C$ be an odd cycle drawn as a standard great circle thrackle. Then $C$ contains at most one long edge.

Following Cairns, Koussas, and Nikolayevsky (2015). First we prove that a good odd cycle can contain at most one long edge. Let $C$ be a good odd $n$-cycle drawing containing $k \geq 2$ long edges. By Lemma 5.13, between every pair of long edges, there is an odd number of short edges. Let $s$ be the number of short edges in $C$. Since $k$ and $s$ have the same parity and $|C| = k + s$, $|C|$ is always even.

In odd cycles containing five or more edges, we claim that no long edges are permitted.

**Lemma 5.16.** Let $C$ be an odd cycle $n$-cycle drawn as a standard great circle thrackle. If $n \geq 5$, then $C$ contains no long edges.

Proof. Now we show that $C$ contains no long edges if $n \geq 5$. Without loss of generality, let $\chi(e_i, e_{i+1}) = 1$. Suppose that $e_1$ is drawn long. We reach a contradiction to this assumption by induction on $n$. Let $n = 5$ and consider $C(e_5)$. Since $C$ is good, $e_1$ emanates from $v_0$ in $H_{e_5}^+$. However, $e_1$ is long, so $v_1$ is in $H_{e_5}^-$. Because $e_2$ crosses $e_5$, $v_2$ is in $H_{e_5}^+$. Similarly, $v_3$ is in $H_{e_5}^-$. Yet since $e_4$ is incident to $v_3$ and $\chi(e_5, e_4) = -1$, $v_3$ should be in $H_{e_5}^+$, as in Figure 52. Therefore, the great circle thrackle of the 5-cycle does not contain a long edge.

Finally suppose that an odd $n$-cycle great circle thrackle cannot be drawn containing a long edge. Let $C$ be an $(n + 2)$-cycle and consider $C(e_{n+2})$. Because the claim holds for $n$, we can arrange the $(n - 1)$-path adjacent to $e_1$ to find that $v_{n-2}$ is in $H_{e_{n+2}}^-$. Then $v_{n-1}$ is in $H_{e_{n+2}}^+$ and $v_n$ is in $H_{e_{n+2}}^-$. Again, since $e_{n+1}$ is incident to $v_n$ and $\chi(e_{n+2}, e_{n+1}) = -1$, $v_n$ should be in $H_{e_{n+2}}^+$. Hence no great circle thrackle of an odd cycle can contain any long edges. □
In their proof of Lemma 5.14, Cairns, Koussas, and Nikolayevsky (2015) show that every great circle thrackle of an even cycle requires at least one long edge. We present an alternate proof in a separate lemma below.

**Lemma 5.17** (Cairns, Koussas, and Nikolayevsky, 2015). *Every good even cycle drawing contains at least one long edge.*

*Proof.* Let us show that every good drawing of an even cycle must contain at least two long edges. Let \( n \geq 6 \) be even and let \( C \) be an \( n \)-cycle. Toward a contradiction, suppose that edges \( e_1, \ldots, e_n \) are short. Then by Lemma 5.12, \( \chi(e_1, e_2) = -\chi(e_n, e_1) \), so \( C \) is not good, which contradicts Lemma 5.14. Therefore, \( C \) requires at least one long edge. \( \square \)

We prove a stronger statement. It is impossible to draw a great circle thrackle of an even cycle with at most two edges. We show below that the lower bound for long edges in a standard great circle thrackle of an even cycle is actually three. Moreover, *every* standard great circle thrackle of an even cycle can be drawn with exactly three long edges.

**Theorem 5.18.** *Every good standard great circle thrackle of an even cycle contains at least three long edges. Moreover, every even cycle can be drawn as a standard great circle*
thrackle with exactly three long edges.

Proof. Let \( n \geq 6 \) be even and let \( C \) be an \( n \)-cycle. Note that the proof of Lemma 5.17 does not depend on whether \( e_1 \) is short or long. If \( e_1 \) is long, then we achieve the same contradiction. Thus, \( C \) cannot contain only one long edge. As in the previous proof, we use contradiction to show that the thrackle of \( C \) cannot contain two edges. Without loss of generality, let \( e_1 \) be drawn long. Suppose that \( e_k \) is the only other long edge. By Lemma 5.12, \( k \) is even. Using Lemmas 5.11 and 5.12 we find that \( \chi(e_2, e_{k+1}) = \chi(e_2, e_n) \). Yet if we consider the path \( e_{k+1} \ldots e_n \), this equality is only possible if \( k + 1 \) and \( n \) have the same parity, by Lemma 5.12. Hence the great circle thrackle of \( C \) cannot contain two edges.

We already have a standard great circle thrackle of the 6-cycle. As a previous example demonstrated, we can obtain the 8-cycle from the 6-cycle through the replacement of a short edge in the 6-cycle with a 3-path containing only short edges. Then for any standard even \( n \)-cycle great circle thrackle drawn with three long edges, performing an edge replacement on a short edge produces a standard \( (n + 2) \)-cycle great circle thrackle. \( \square \)

5.1.3 General Results on Thrackleable Graphs

All of the following lemmas apply broadly to standard great circle thrackles. We begin this section with the notions of reaching vertices through a hemisphere and separation at a vertex.

Definition 5.19. Edge \( e \) reaches vertex \( v \) through hemisphere \( H \) if \( e \) is incident to \( v \) and in a small neighborhood of \( v \), the interior of \( e \) is in \( H \).

In Figure 53, \( f \) reaches \( v_2 \) through \( H_{e_1}^+ \) and \( g \) reaches \( v_1 \) through \( H_{e_1}^- \).
Definition 5.20. Let $v \in G$ with $\deg(v) \geq 3$. Edge $e$ separates at $v$ if there exist edges $f, g \in G$ incident to $v$ such that $f$ reaches $v$ through $\mathcal{H}_{e}^{+}$ and $g$ reaches $v$ through $\mathcal{H}_{e}^{-}$.

Using this definition, we present the following lemma.

Lemma 5.21 (Cairns, Koussas, and Nikolayevsky, 2015). Every edge that separates at a vertex in a standard great circle thrackle is short and is an edge of a bad 3-cycle.
Proof. Let \( e, f, \) and \( g \) be adjacent edges in \( G \) drawn so that \( e \) separates at their shared vertex \( v \). Without loss of generality, let \( f \in \mathcal{H}_e^+, g \in \mathcal{H}_e^-, \) and \( \chi(e, f) = -1 \). Then \( \chi(e, g) = 1 \). Let \( v' \) be the vertex adjacent to \( v \) via \( e \). Because \( G \) is drawn as a standard great circle thrackle, it contains no leaves. Therefore, there must be some edge \( h \in G \) adjacent to \( e \) and incident to \( v' \). If \( \chi(h, e) = 1 \) \((-1) \), then \( hfe \) \(( heg) \) is a bad 3-path. Thus, by Lemma 5.7, \( e \) is short, either \( h \) or \( f \) \(( g) \) is long, and \( e \) is an edge in a bad 3-cycle.

Figure 55: Diagram for Lemma 5.21

The following two lemmas deal with limits on the degree of vertices in standard great circle thrackleable graphs.

**Lemma 5.22** (Cairns, Koussas, and Nikolayevsky, 2015). Let \( G \) be a standard great circle thrackle. For any vertex \( v \in G \) with \( \deg(v) \geq 3 \), there is an edge \( e \) incident to \( v \) such that every other edge incident to \( v \) reaches \( v \) through the same hemisphere bounded by \( C(e) \).

**Proof.** Toward a contradiction, suppose that no such edge \( e \) exists. Then every edge incident to \( v \) separates at \( v \). Therefore, every edge incident to \( v \) is a short middle
edge of a bad 3-path by Lemma 5.21, so $G$ must contain at least two 3-cycles, which contradicts Theorem 3.7.

**Lemma 5.23** (Cairns, Koussas, and Nikolayevsky, 2015). *Every vertex in a standard great circle thrackleable graph $G$ has at most degree four. If a vertex, $v$, has degree three or greater, then $v$ is a vertex of a bad 3-cycle.*

*Proof.* We reach a contradiction by supposing that for some $v \in G$, $\deg(v) = k > 4$. By Lemma 5.22, there is some edge $e$ incident to $v$ such that the other $k - 1$ edges incident to $v$ reach $v$ through the same hemisphere of $C(e)$. This configuration implies that at least $k - 2$ edges incident to $v$ separate at $v$. By Lemma 5.8 and Lemma 5.21, $G$ contains more than one 3-cycle. Hence $\deg(v) \leq 4$. □

### 5.1.4 The Thrackle Conjecture on the Sphere

We turn now to proving that the Thrackle Conjecture holds for great circle thrackleable graphs. First we need the following lemma about potential counterexamples.

**Lemma 5.24** (Cairns, Koussas, and Nikolayevsky, 2015). *Suppose that $G$ is a standard great circle thrackleable graph that is a minimal counterexample to the Thrackle Conjecture. Then $G$ is neither a dumbbell nor a theta graph.*

*Proof.* By Theorem 4.7, $G$ is either a dumbbell, a theta graph, or a figure-8. Furthermore, since $G$ defies the Thrackle Conjecture, it must contain at least one vertex of degree 3 or more. By Lemma 5.23, any such vertex is contained in a bad 3-cycle. Thus, by Theorem 3.7, $G$ contains exactly one 3-cycle. Call this 3-cycle $C$. Since no vertex disjoint from $C$ has degree greater than 2, $G$ is not a dumbbell.

Toward a contradiction, suppose that $G$ is a theta graph. Then $G$ contains an outer cycle $H$ and two inner cycles, $D$ and the 3-cycle $C$ described above. Note that $D$ and $H$ have different parities. Suppose that $D$ is an even cycle. Then $D$ must contain at least three long edges, by Lemma 5.18. At least one of these edges must be shared with $H$, an odd cycle, which contradicts Lemma 5.16. On the other hand, if $H$ is an even cycle,
then it must share a long edge with \( D \), an odd cycle, which again contradicts Lemma \[ \text{5.16} \] Hence \( G \) is neither a dumbbell nor a theta graph.

We are left with just a short proof that the Thrackle Conjecture is true for great circle thrackles on the sphere.

**Theorem 5.25** (Cairns, Koussas, and Nikolayevsky, 2015). Let \( G \) be a standard great circle thrackleable graph. Then \( G \) contains at most as many edges as vertices.

*Proof.* Toward a contradiction, suppose that \( G \) contains more edges than vertices. By Lemmas \[ \text{4.7} \] and \[ \text{5.24} \] \( G \) contains a figure-8 subgraph, \( A \). Let \( C \) be the bad 3-cycle subgraph of \( A \), as described in the proof of Lemma \[ \text{5.24} \] and \( D \) the other cycle subgraph of \( A \). By Lemma \[ \text{3.6} \] \( D \) is even; thus \( D \) must contain at least one long edge (Lemma \[ \text{5.14} \]). Let the edges of \( C \) be labeled \( e_1 \), \( e_2 \), and \( e_3 \), with \( e_2 \) being long. Note that any directed walk excluding \( e_2 \) of \( C \) is good (see Figure 56). Otherwise, such a walk would contain a bad 3-path, which would require a second bad 3-cycle by Lemma \[ \text{5.8} \]. Thus there is a good path from any edge in \( D \) to \( e_1 \) or \( e_3 \). Denote the number of long edges in \( D \) by \( k \). Suppose that the directed path of short edges between the terminal vertex of \( e_2 \) and the first long edge in \( D \) has odd length. Then, because \( D \) is an even cycle, the path of short edges between the \( k \)th long edge in \( D \) and the initial vertex of \( e_2 \) has even length, which contradicts Lemma \[ \text{5.13} \]. Hence \( G \) cannot contain a figure-8 subgraph and has at most as many edges as vertices.

![Figure 56: Possible Configurations of Subgraph A of G](image)

\[ \quad \]

\[ \square \]
Now we have shown that standard great circle thrackles obey the Thrackle Conjecture. Because standard great circle thrackleable graphs contain no leaves, this label applies only to thrackleable cycles. Moreover, it is not difficult to see that standard great circle thrackleable graphs are a subset of a larger class of graphs that can be drawn as thrackles using the arcs of great circles. We discuss that greater class in the following section. This exercise is of particular interest, because if every thrackleable graph is great circle thrackleable, then the Thrackle Conjecture must be true.

5.1.5 Classifying Great-circle Thrackleable Graphs

Although trees have been excluded from the discussion so far, we know that at least some of them can be drawn as thrackles using arcs of great circles to represent their edges. In particular, all straight-line thrackleable graphs are great-circle thrackleable. For the remainder of the section, let us use the term great-circle thrackle distinctly from standard great-circle thrackle.

Definition 5.26. A great-circle thrackle is a thrackle of a connected graph, drawn on the sphere in general position, whose edges are represented by arcs of great circles.

Recall that a great-circle thrackle is in general position if no three vertices are drawn in the same great circle and no two vertices are represented by antipodal points. We do not alter the definition of good and bad $k$-paths and $k$-cycles. In fact, the only characteristic of standard great-circle thrackleable graphs that we remove is the prohibition on leaves.

Whereas Woodall (1971) showed that the trees that are straight-line thrackleable are limited to caterpillars in Theorem 4.21, we claim that a greater class of trees can be thrackled using arcs of great circles. At the same time, we claim that there are trees that cannot be drawn as great circle thrackles.

Note that several results from the previous section are relevant to a great-circle thrackleable graphs that are non-standard. In particular, Lemmas 5.6, 5.7, 5.10, 5.11.
and 5.12 do not depend on the absence of leaves; certainly, Lemmas 5.13, 5.14, 5.18 apply to cycle subgraphs of great circle thrackleable graphs.

Although in this section we present mostly results that apply specifically to great-circle thrackleable trees, we have immediate results for fuzzy even cycles. Recall that a fuzzy $n$-cycle is a graph $G$ containing an $n$-cycle subgraph $C$ so that every vertex in $G$ is adjacent to a vertex of $C$.

**Theorem 5.27.** Every fuzzy even cycle is great circle thrackleable.

*Proof.* Let $G$ be an $n$-cycle, where $n \geq 6$. Let $U_i$ be a small neighborhood of edge $e_i$ and $V_j$ a small neighborhood of vertex $v_j$, for all $v_j \in V(G)$ and $e_i \in E(G)$. Without loss of generality, let long edges have even indices.

Suppose that $n = 6$. For even $i$, fix point $p_i$ in $V_{i+1} \cap int(e_{i+2})$. Draw a great circle arc from $v_i$ through $U_{i+1}$ and $p_i$ to some point $w_i$ in $V_{i+1}$. This arc $f_i = \{v_i, w_i\}$ crosses all edges that $e_{i+1}$ crosses and $f_i$ crosses $e_{i+2}$ at $p_i$.

For odd $i$, fix point $p_i$ in $V_{i-1} \cap int(e_{i-1})$. Draw a great circle arc from $v_i$ through $U_i$ and $p_i$ to some point $w_i$ in $V_{i-1}$. Again, this arc $f_i$ crosses all edges that $e_i$ crosses and crosses $e_{i-1}$ at $p_i$. Note that since many $p_i$ are possible and great circles between pairs of non-antipodal points are unique, there is no upper bound on the number of edges that can be attached in this way to each vertex of the cycle subgraph.

This drawing is a great circle thrackle. See Figures 57 and 58 for reference.
Now suppose that for some \( n > 6 \), every fuzzy \( n \)-cycle is great circle thrackleable. Perform an edge replacement on a short edge of the cycle subgraph to obtain a fuzzy \( (n + 2) \)-cycle containing degree-2 vertices. It suffices to show that the degree-2 vertices can have leaves attached via arcs of great circles.
For the new vertex each new \(v_i\) in our graph, fix point \(p_i\) in \(V_{i+1} \cap \text{int}(e_{i+2})\). Draw a great circle arc from \(v_i\) through \(U_{i+1}\) and \(p_i\) to some point \(w_i\) in \(V_{i+1}\). This arc \(f_i = \{v_i, w_i\}\) crosses all edges that \(e_{i+1}\) crosses and \(f_i\) crosses \(e_{i+2}\) at \(p_i\).

We include an illustration of a fuzzy 8-cycle below.

![Figure 59: Fuzzy 8-cycle](image)

As we turn to great circle thrackleable graphs that include trees, the concept of a pointed vertex is useful in this discussion. We update Definition 4.20 for spheres.

**Definition 5.28.** If \(v\) is a pointed vertex in a graph drawing on the sphere, then all of the edges incident to \(v\) reach \(v\) through a single hemisphere whose boundary is a great circle through \(v\).

We can use this definition to characterize edges that separate at \(t\) vertex. This definition leads immediately to the following lemma.
Lemma 5.29. Let $G$ be a great-circle thrackleable graph and let $\deg(v) = k$ for some $v$ in $G$. Then at least $k - 2$ edges separate at $v$. If $v$ is pointed, then exactly $k - 2$ edges separate at $v$.

Proof. Label the edges incident to $v$ $e_1, \ldots, e_k$. Let $l_i$ be the length of edge $e_i$ and let $U$ be the circular neighborhood of radius $r$, $0 < r < \min\{l_i\}_{1 \leq i \leq k}$, centered at $v$.

Suppose that $k = 3$ and that neither $e_1$ nor $e_2$ separates at $v$. Then $e_2$ and $e_3$ reach $v$ through the same hemisphere of $C(e_1)$, say $H_{e_1}^+$, and $e_1$ and $e_3$ reach $v$ through the same hemisphere of $C(e_2)$, say $H_{e_2}^+$. Thus $e_3$ reaches $v$ through $A = U \cap (H_{e_1}^+ \cap H_{e_2}^+)$, which is bounded in part by $e_1$ and $e_2$. Because neither $e_1$ nor $e_2$ separates at $v$, the angle between $e_1$ and $e_2$ in $A$ is less than $\pi$. Moreover, $C(e_3)$ passes through the interior of $A$, so $e_1$ and $e_2$ reach $v$ through opposite hemispheres of $C(e_3)$.

Now suppose that $k = n$ and that the claim holds if one edge incident to $v$ is removed. Call this removed edge $e_n$. If either all or all but one edge incident to $v$ separates at $v$, then the re-insertion of $e_n$ in $G$ fulfills the claim. We complete the proof by assuming that two edges in $G \setminus e_n$ do not separate at $v$. Denote these edges $e_j$ and $e_l$, where $j, l \in [1, \ldots, n - 1]$. Re-insert $e_n$ and consider the subgraph $(e_j \cap e_l \cap e_n)$ of $G$. By the base case, one of the edges $e_j, e_l, e_n$ separates at $v$.

Suppose that $v$ is pointed. Denote by $H_v$ the hemisphere through $v$ through which all edges incident to $v$ reach $v$. Let $C(v)$ be the boundary of $H_v$. Let $e_1$ and $e_k$ be the edges drawn at the smallest positive angle from $C(v)$. Then edges $e_2, \ldots, e_k$ reach $v$ in the portion of $H_v$ bounded by $C(v)$ and $C(e_1)$. Similarly, edges $e_1, \ldots, e_{k-1}$ reach $v$ in the portion of $H_v$ bounded by $C(v)$ and $C(e_k)$. Thus exactly $k - 2$ edges separate at $v$. □
Let us characterize the edges that separate at some vertex.

**Lemma 5.30.** Every long edge that separates at a vertex is terminal.

*Proof.* Let $G$ be a great circle thrackleable graph. Let $v_0 \in G$ be a vertex of degree $k \geq 3$ and $e_1$ an edge that separates at $v$. Let $f$ and $g$ be adjacent to $e$ via $v_0$. Suppose that $e_1$ is drawn long. Toward a contradiction, suppose that $e_2$ is adjacent to $e_1$ via $v_1$. Since $f$ and $g$ are in different hemispheres of $C(e_1)$, $e_2$ must be long to cross them both. Yet this is not possible, by Lemma 5.6. □

We now describe spiders to find potential contradictions to Woodall’s Theorem among great-circle thrackleable trees. Recall that a spider is a tree in which exactly one vertex, $v_0$ has degree at least three and all others have degree at most two. First we show that there is no upper bound on the number of legs in a spider.

**Theorem 5.31.** A finite spider on $k \geq 3$ legs of length one is great-circle thrackleable.

*Proof.* Let $G$ be a finite spider on $k$ legs of length one and let $v$ be the degree-$k$ vertex in $G$. Fix $v$ on the sphere and fix an orientation about $v$. Label the edges of $G$ $e_1, e_2, \ldots, e_k$. Draw edge $e_{i+1}$ so that it forms the angle $\frac{2\pi}{k}$ with edge $e_i$ for $1 \leq i < k$. If all edges are drawn short, this description suffices. Otherwise, suppose that some $e_j$ is drawn long. By Lemma 5.6 this is the only long edge in this drawing of $G$. Then draw all remaining edges short as described above. The resulting drawing is a great-circle thrackle. □

Note that in the theorem above, if $v$ is pointed, it suffices to reduce the angle between successive edges. To complete this section, we extend the notion of separation to paths.

**Definition 5.32.** A $k$-path $P = e_1 \ldots e_k$ separates at vertex $v$ if either $e_1$ or $e_k$ separates at $v$.

This term allows us to exclude some graphs from the class of great circle thrackleable graphs.
Lemma 5.33. Let $G$ be a great circle thrackleable graph and let $v \in G$ be a vertex of degree $k \geq 3$. If all edges incident to $v$ are short, then any path that separates at $v$ contains at most two edges.

Proof. Let path $P$ separate at degree-$k$ vertex $v_0$, with $e_1 \subseteq P$ incident to $v_0$. Suppose that the length of $P$ is greater than one. Since $e_1$ separates at $v_0$, there exist some $f$ and $g$ that reach $v_0$ through opposite hemispheres of $C(e_1)$. By Lemma 5.30, $e_1$ is drawn short. Because $f$ and $g$ are in opposite hemispheres of $C(e_1)$ and $v_1$ (one of the endpoints of $e_2$) lies on $C(e_1)$, $e_2$ must be long to cross both $f$ and $g$. Without loss of generality, suppose that $e_2$ reaches $v_1$ through the hemisphere if $C(e_1)$ on which $f$ reaches $v_0$, say $H_{e_1}(f)$. Then $v_2$ lies in $H_{e_1}(g)$, between $g$ and $e_1$.

Now suppose toward a contradiction that $P$ contains a third edge, $e_3$. Because $v_2$ lies in $H_{e_1}(g)$, but $e_3$ must cross both $f$ and $g$, $e_3$ must be long, which is not possible by Lemma 5.6.

We are now ready to demonstrate that there is a tree counterexample to Woodall's Theorem (Theorem 2.1). If the spider on three legs of length two can be drawn as a great-circle thrackle, then the class of admissible trees includes more than caterpillars. In fact, that is the case.

Theorem 5.34. The spider on three legs of length two is great circle thrackleable.

Proof. Let us construct $G$, a spider on three legs, each of length two. Let $v$ be its degree-$3$ vertex and let $v$ be pointed. Fix a hemisphere containing containing all edges incident to $v$ and call it $S$. Fix a short edge in $S$ and call it $e_1$. Let $e_1$ be an edge in path $P$. Suppose that $P$ contains a second edge, $e_2$, so that $e_2$ is long and $P$ is drawn in general position. Because $C(e_2)$ is a great circle and since we have the assumption of general position, every point on $C(e_2)$ has distance less than $\pi$ from $v$. Let $\max_{e_2}$ be the maximum distance from $v$ to any point on $C(e_2)$ within $S$ and let $\min_{e_2}$ be the minimum distance from $v$ to any point on $C(e_2)$ within $S$. 71
Draw $f_1$ in $\mathcal{H}_{e_1}^+ \cap S$ so that it crosses $e_2$ with length $l_{f_1}$ greater than $max_{e_2}$, but less than $\pi$. Similarly, draw $g_1$ in $\mathcal{H}_{e_1}^- \cap S$ so that it crosses $e_2$ with length $l_{g_1}$ greater than $max_{e_2}$, but less than $\pi$. Let $w_f$ be the endpoint of $f_1$ and $w_g$ the endpoint of $g_1$. So far, this graph is a spider on three legs, two of which have length one. It is also a caterpillar.

Let $p_f$ be a point on $f_1$ with positive length less than $min_{e_2}$ and $p_g$ a point on $g_1$ with positive length less than $min_{e_2}$. Draw $f_2$ as a short arc on the great circle containing $w_f$ and $p_g$ so that $p_g$ is in the interior of $f_2$. Draw $g_2$ similarly.

This drawing is a spider on three legs, each of length two.

Remark. We do not wish to claim that $f_1$, $f_2$, $g_1$, and $g_2$ must be short in the proof above. Neither must the separating edge always be shorter than the non-separating edges. Making these assumptions facilitates the proof, but we do not claim that they are requirements.

Because the spider on three legs of length two is great circle thrackleable, we know that the class of trees that are thrackleable using great circles is larger than caterpillars. Yet Theorem 5.33 demonstrates that there are limits to the trees that are thrackleable in this way. We define augmented caterpillars, a class of trees which includes those that are thrackleable using great circles.

Definition 5.35. An augmented caterpillar, $G$ is a tree consisting of a longest path, its spine, and paths terminating at internal vertices of the spine, its legs. Every vertex in $G$ is maximum distance two from an internal vertex of $G$’s spine.

To demonstrate that every great circle thrackleable tree must be contained in the class of augmented caterpillars, it suffices to show that the spider on three legs of length three (Figure 60) is not great circle thrackleable. Lemma 5.33 provides some evidence of this result. We make the statement explicit below.
Theorem 5.36. The spider on three legs of length three cannot be drawn as a great circle thrackle.

Proof. Let $G$ be a spider on three legs of length three. Let $z \in G$ be the degree-3 vertex. Label the legs of $G$ $E = e_1e_2e_3$, $F = f_1f_2f_3$, and $H = h_1h_2h_3$. Let the initial edges in $E$ be $e_1 = \{z, v_1\}$; in $F$, $f_1 = \{z, w_1\}$; and in $H$, $h_1 = \{z, u_1\}$. Let the remaining edges be $e_i = \{v_{i-1}, v_i\}$, $f_i = \{w_{i-1}, w_i\}$, and $h_i = \{u_{i-1}, u_i\}$ for $i = 2, 3$, as in Figure 60.

Suppose, toward a contradiction, that $G$ can be drawn as a great circle thrackle. By Lemmas 5.6 and 5.34, one of $e_1$, $f_1$, and $h_1$ is long. Without loss of generality, let $e_1$ be long. By Lemma 5.30, $e_1$ does not separate at $z$. Again without loss of generality, let $f_1$ separate at $z$, as in Figure 61. Denote by $l_{e_1}$, $l_{f_1}$, and $l_{h_1}$ be the lengths of $e_1$, $f_1$, and $h_1$, respectively. To admit a great circle arc containing $w_1$ and interior points of $e_1$ and $h_1$, let $l_{e_1} > \pi + \max\{l_{f_1}, l_{h_1}\}$. This condition is necessary, but not sufficient to demonstrate that $f_2$ can be drawn to cross both $e_1$ and $h_1$, so we further require that $l_{h_1} > l_{f_1}$. 

Figure 60: Spider on Three Legs of Length 3
Although $e_1$ and $h_1$ reach $z$ through different hemispheres of $\mathcal{C}(f_1)$, the segment of $e_1$ bounded by $v_1$ and the antipodal point of $z$ is in the same hemisphere of $\mathcal{C}(f_1)$ as $h_1$.

Draw a short great circle arc $f_2$ from $w_1$ through $h_1$ and $e_1$, terminating at $w_2$. Let $p_1$ be the point where $f_2$ crosses $h_1$ and $q_1$ the point where $f_2$ crosses $e_1$, as in Figure 62.
Denote by $q_2$ an interior point of $e_1$ on the arc bounded by $q_1$ and $v_1$. Draw short great circle arc $f_3$ from $w_2$ through $q_2$, terminating at $w_3$, so that $f_3$ crosses $h_1$ and $e_1$ as in Figure 63. This process completes $F$.

![Figure 63: Spider Subgraph of $G$ Containing Two Legs of Length One, One Leg of Length Three](image)

Note that the arcs $[z, w_1]$, $[w_1, p_1]$, and $[p_1, z]$ form a 3-cycle. Since $v_1$ lies outside this 3-cycle, any great circle arc containing $v_1$ and crossing $f_1$, $f_2$, and $g_1$ must contain $p_1$. Suppose that $e_2$ contains both $p_1$ and an interior point of $f_1$. Then $e_2$ crosses $f_1$ in one hemisphere of $f_2$ and meets $e_1$ in the other hemisphere. In other words, $q_1$ lies between $v_1$ and the intersection of $e_1$ and $e_2$. This situation is impossible, because the intersection of $e_1$ and $e_2$ is $v_1$ as shown in Figure 64.

Hence the spider on three legs of length three is not great circle thrackleable. □
Theorem 5.37. If a tree \( G \) is great circle thrackleable, then \( G \) is an augmented caterpillar.

Proof. As we have mentioned, Woodall’s Theorem (2.1) gives us the result that all caterpillars are great circle thrackleable. The remainder of the proof is immediate from Lemma 5.33, Theorem 5.34, and Theorem 5.36.

Unfortunately, we have not elaborated enough to claim that if a graph is an augmented caterpillar, then it is great circle thrackleable. We leave the following two conjectures to be proved.

Conjecture 5.38. The class of great circle thrackleable trees is properly contained in the class of augmented caterpillars.

Conjecture 5.39. Let \( G \) be an augmented caterpillar drawn as a a great circle thrackle and let \( v \) and \( v' \) be adjacent in the spine of \( G \). If some 2-path separates at \( v \), then no 2-path separates at \( v' \).

Furthermore, it remains to be shown how the class of fuzzy \( n \)-cycles can be expanded to include all great circle thrackleable graphs containing \( n \)-cycles.
5.2 Thrackles on Surfaces of Positive Genus

In this section we consider graphs that are thrackleable on surfaces of positive genus. We note that the class of graphs that are thrackleable on surfaces of positive genus is larger than the class of graphs that are thrackleable in the plane. In particular, the Thrackle Conjecture does not hold on surfaces of positive genus.

**Definition 5.40.** An embedding is an injective map, \( f \), that is a homeomorphism with the image of \( f \), \( \text{im}(f) \).

We can embed graphs that are thrackleable in the plane to obtain thrackles on multiple surfaces. In the case of surfaces with positive genus, Woodall (1971) demonstrated that we can embed some graphs as thrackles, even if they are not thrackleable in the plane. For example, the 4-cycle is thrackleable on the torus, as in Figure 65.

![Figure 65: 4-Cycle Thrackle on the Torus](image)

In fact, Woodall also showed that the figure-8 containing two 4-cycles is thrackleable on the torus, as in Figure 66.
Thus, we see that the Thrackle Conjecture does not hold on the torus. It follows immediately that the Thrackle Conjecture does not hold on any orientable surface of positive genus. This result gives rise to new upper bounds for edges in a thrackle based on the work of Lovasz, Pach, and Szegedy (1997). Notably, Cairns and Nikolaevsky (2000) have proved the following theorem.

**Theorem 5.41** (Cairns and Nikolaevsky, 2000). Let $G$ be a thrackleable graph on $n$ vertices that contains an odd cycle $C_1$, which intersects every other cycle $C_2 \subseteq G$ an even number of times. Suppose that $G$ is thrackleable on surfaces of genus 1. Then $G$ contains at most $2n - 2$ edges.

**Proof.** Construct $G'$ as in Theorem 3.14. Note that four cycles are thrackleable on the torus. Thus, using Euler's Formula, we obtain

$$2m' \geq 4(f - 1) + 2k_1.$$  \hspace{1cm} (6)

Therefore,

$$m \leq 2n - 2.$$  \hspace{1cm} (7)

\hfill \Box
In addition, researchers have examined graph drawings that are close to thrackles but not thrackles. Because the main question of the Thrackle Conjecture is answered on the torus, some researchers have begun to generalize the results on thrackles in the plane to *generalized thrackles*.

**Definition 5.42** (Woodall, 1971). A *generalized thrackle* is a graph drawing in which every pair of edges meets an odd number of times.

Archdeacon and Stor (2017) have researched a variant of the thrackle called a *superthrackle*.

**Definition 5.43** (Archdeacon and Stor, 2017). A *superthrackle* is a graph drawing in which every pair of edges crosses exactly once.

Note that the main difference between a thrackle and a superthrackle is that adjacent edges must cross in a generalized thrackle.

As the continuing research about thrackles shows, there continue to be questions about this sort type of graph drawing, and more general questions on surfaces. We categorize the thrackles discussed in this section using the pseudo-taxonomy in Figure 67.

Figure 67: A Pseudo-taxonomy of Thrackle Types
6 Conclusion

As we have shown in the preceding sections, the Thrackle Conjecture is a complicated idea, even if it stated simply. Multiple authors have attempted to prove it, using diverse methods. Yet none has yet demonstrated that it is either true or not. What has emerged from the research on thrackles is a classification of thrackleable graphs that did not previously exist. Figure 68 demonstrates this organization. In addition, the appendix contains a catalogue of some of the known thrackleable graphs.

While there has been significant progress in finding upper bounds on the edges in thrackleable graphs, we can see quite clearly that that progress has slowed significantly. Other innovative approaches have been promising, but not one has yet made a definitive statement about the Thrackle Conjecture. Prof. Conway left behind a tangled line of inquiry, indeed.
Figure 68: A Pseudo-taxonomy of Thrackle Types
Appendix: Connected Thrackleable Graphs on $n$ Vertices

In this appendix we present a list of thrackleable graphs on up to eleven vertices. We chose this upper bound because Wehner (2010) claims that the Thrackle Conjecture is true for graphs on up to eleven vertices. We are confident that the list of graphs on up to six vertices is complete, having relied on previous research, notably Cvetic and Petric (1984). We hope that the list of graphs on up to eight vertices is complete. We claim that it includes all the thrackleable graphs in the list of small graphs in the Information System on Graph Classes and Their Inclusions (de Ridder, 2021). Note that the ISGCI is a list of graph classes, not necessarily graphs themselves. In any case, we would be grateful for any independent confirmation. To compile the graphs on nine or more vertices, the author relied on the ISGCI and some crude, self-written Python code, included at the end of the Appendix. This sublist is very incomplete.

2 Vertices

\[
\begin{array}{c}
\text{\(v_0\)} \\
\text{\(v_1\)} \\
\text{\(v_0\)}
\end{array}
\]

3 Vertices

\[
\begin{array}{c}
\text{\(v_1\)} \\
\text{\(v_2\)} \\
\text{\(v_0\)}
\end{array}
\]

\[
\begin{array}{c}
\text{\(v_0\)} \\
\text{\(v_1\)} \\
\text{\(v_2\)}
\end{array}
\]
6 Vertices
7 Vertices
8 Vertices
9 Vertices
10 Vertices
11 Vertices

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

v_0 v_1 v_2 v_3 v_4
v_5 v_6 v_7 v_8 v_9
v_10

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Python Code Used to Compile Some Graphs

```python
# C. Roberts
# Graphs on 9+ vertices
# 2021-01-26

import math

pi=3.14159

### GENERATE TIKZ CODE FOR THRACKLE ###

# check to see if in degrees or radians
print('If this value is 1, then good to go:', math.sin(pi/2))
print('If this value is 0, then good to go:', math.sin(pi))

def cycle(n):
    # 1 + int --> NoneType
    if n>2:
        print('n, cycle)
        print(\begin{center} \begin{tikzpicture} \end{tikzpicture} \end{center})
        for x in range(0,n):
            print(\filldraw ('\',\textcolor{black}{\text{math.cos(x*2*pi/n)},','},\textcolor{black}{\text{math.sin(x*2*pi/n)},'}) circle (2.5pt) node[below left:{$\nu_{['},x,']}\end{tikzpicture} \end{center})
            for x in range(0,n):
                print('draw(\',\textcolor{black}{\text{math.cos(x*2*pi/n)},','},\textcolor{black}{\text{math.sin(x*2*pi/n)},}')--('\',\textcolor{black}{\text{math.cos(x+1)*2*pi/n)},','},\textcolor{black}{\text{math.sin(x+1)*2*pi/n)},')\end{tikzpicture} \end{center})
        print('}
        print('}'
```
def cyclepath(n):
    """sig int --&gt; NoneType
    for y in range(1,n-2):
        if n-y==4:
            print('"""y',n-y,'-cycle with',y,'-path attached to 1 vertex"
            print('\\begin{center} \begin{tikzpicture}"
            for x in range(0,n-y):
                print('\\filldraw (',math.cos(x*2*pi/(n-y)),',',math.sin(x*2*pi/(n-y)),') circle (2.5pt) node\{label=below:{$y$$_{'}x',1}'\};"
            for x in range(0,n-y):
                print('\\filldraw (',x-n+y+2,',0) circle (2.5pt) node\{label=below:{$y$$_{'}x',1}'\};"
            for x in range(0,n-y):
                print('\\draw',math.cos(x*2*pi/(n-y)),',',math.sin(x*2*pi/(n-y)),')--(',math.cos((x+1)*2*pi/(n-y)),',',math.sin((x+1)*2*pi/(n-y)),');"
            for x in range(1,n-y):
                print('\\draw(',x,',0)--(',x+1,',0);"
            print('\\end{tikzpicture} \end{center}"
            print('"
else:
    print('"""Not possible"
    print('"""
def cyclefork(n):
    # sig int -->> NoneType
    for y in range(1, n-2):
        if (n-y-1>y) & (n-y-1<n):
            print("'n", n-y-1, 'cycle with', y-1, 'path, that ends in a fork, attached to 1 vertex'
        print('\n\begin{center} \begin{tikzpicture}\n        for x in range(0, n-y-1):
            print('\n\filldraw ('\math.cos(x*2*pi/(n-y-1)), ',', 'math.sin(x*2*pi/(n-y-1)), ') circle (2.5pt) node\label{below:('')};
        for x in range(n-y,n):
            print('\n\filldraw (''x+\pi/2', ',', '0) circle (2.5pt) node\label{below:('')};
        print('\n\filldraw (''y+1', ',', '6) circle (2.5pt) node\label{below:('')};
        print('\n\filldraw (''y-1', ',', '-6) circle (2.5pt) node\label{below:('')};
        for x in range(0, n-y-1):
            print('\n\draw(', 'math.cos(x*2*pi/(n-y-1)), ', ', 'math.sin(x*2*pi/(n-y-1)), ')--(', 'math.cos((x-1)*2*pi/(n-y-1)), ', ', 'math.sin((x-1)*2*pi/(n-y-1)), ');
        for x in range(1,y):
            print('\n\draw(', 'x, ', '0)--(', 'x+1, '0);
        print('\n\draw(', 'y, ', '0)--(', 'y+1, '0);
        print('\n\draw(', 'y, ', '0)--(', 'y+1, ', '-6);
        print('\n\end{tikzpicture} \end{center}\n        print(''
    else:
        print(''	Not possible'
        print(''

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```python
def cycle_tree(n):
    # siu int --> NoneType
    for y in range(1,n-1):
        if (n-y-1>2):
            print('\text{'n-y-1'=''}\times(n-y-1)'\quad\text{cycle with}'\quad'y', 'path, that ends in a fork, attached to 1 vertex'}\)
            print('\begin{center} \begin{tikzpicture}''
            for x in range(0,n-2):
                print('\filldraw (',math.cos(x*2*pi/(n-y-1)),',',math.sin(x*2*pi/(n-y-1)),',') circle (2.5pt) node[label=below:($v_{('',x,'')}$)];'')
            for x in range(n-y,2):
                print('\filldraw (',x+n-y-2,',0) circle (2.5pt) node[label=below:($v_{('',x,'')}$)];'')
            for x in range(n-y+2,6):
                print('\filldraw (',y+1,',0) circle (2.5pt) node[label=below:($v_{('',y+1,'')}$)];'')
            for x in range(n-y+2,6):
                print('\filldraw (',y-2,',0) circle (2.5pt) node[label=below:($v_{('',y-2,'')}$)];'')
            for x in range(n-y+2,6):
                print('\filldraw (',y-1,',0) circle (2.5pt) node[label=below:($v_{('',y-1,'')}$)];'')
            if x in range(1,6):
                print('\draw (',y-1,',0)--(',y+1,',0);\)')
            else:
                print('\draw (',y-1,',0)--(',y+1,',0);\)')
            print('\end{tikzpicture} \end{center}')
            print('')

def path(n):
    # siu int --> NoneType
    if n>1:
        print('\text{'n-3'=''}\times(n-3)'\quad\text{path')
        print('\begin{center} \begin{tikzpicture}''
        for x in range(0,n):
            print('\filldraw (',x+',0) circle (2.5pt) node[label=below:($v_{('',x,'')}$)];'')
        for x in range(1,n-1):
            print('\draw (',x+',0)--(',x+1,',0);\)')
        else:
            print('\draw (',x+',0)--(',x+1,',0);\)')
        print('\end{tikzpicture} \end{center}')
        print('')
```
```python
def fork(n):
    # sig int --> NoneType
    if n>2:
        print("'r',n-2,'-path, ending in a fork'")
        print("\begin{center} \begin{tikzpicture}\)
        for x in range(0,n-2):
            print("\filldraw ('r',x,0) circle (2.5pt) node[label=below:$v_{x'}$,right:$v_{n-1}$)];")
        for y in range(0,n):
            print("\draw(',x,0)--(',x+1,0);")
        print("\draw(',n-3,0)--(',n-2,0);")
        print("\draw(',n-3,0)--(',n-2,0);")
        print("\end{tikzpicture} \end{center}\)
        print('')
    else:
        print("'no fork '")
        print('')

def star(n):
    # sig int --> NoneType
    if n>3:
        print("'r',n-1,'-star'")
        print("\begin{center} \begin{tikzpicture}\)
        print("\filldraw (0,0) circle (2.5pt) node[label=below:$v_0$,:];")
        for x in range(1,n):
            print("\filldraw ('r',math.cos(x*2*pi/(n-1)),',',math.sin(x*2*pi/(n-1)),',') circle (2.5pt) node[label=below:$v_{x'}$,right:$v_{x'}'];")
        for x in range(0,n):
            print("\draw(',math.cos(x*2*pi/(n-1)),',',math.sin(x*2*pi/(n-1)),',')--(0,0);")
        print("\end{tikzpicture} \end{center}\)
        print('')

for y in range(9,12):
    cycle(y)
cyclepath(y)
cyclefork(y)
path(y)
fork(y)
star(y)
```
References


